

# Characteristic Functions, Liftings and Modules

## Habilitationsschrift

zur Erlangung des akademischen Grades

doctor rerum naturalium habilitatus

(Dr. rer. nat. habil.)

an der

Mathematisch-Naturwissenschaftlichen Fakultät

der Ernst-Moritz-Arndt-Universität Greifswald

vorgelegt von Santanu Dey

geboren am 18. März 1976

in Bankura

Greifswald, 2009



# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>7</b>
2.1	Minimal isometric dilations . . . . .	7
2.2	Theorems of Stinespring and Beurling . . . . .	10
2.3	Characteristic function of Popescu . . . . .	14
2.4	Hilbert $C^*$ -module . . . . .	19
<b>3</b>	<b>Characteristic Functions for Ergodic Tuples</b>	<b>25</b>
3.0	Introduction . . . . .	26
3.1	Weak Markov dilations and conjugacy . . . . .	28
3.2	Ergodic coisometric row contractions . . . . .	33
3.3	A new characteristic function . . . . .	38
3.4	Explicit computation . . . . .	43
3.5	Case II is Popescu's characteristic function . . . . .	46
3.6	A complete unitary invariant . . . . .	49
3.7	Example . . . . .	52
3.8	Appendix . . . . .	55
<b>4</b>	<b>Characteristic Functions of Liftings</b>	<b>59</b>
4.1	Subisometric liftings . . . . .	63

4.2	Coisometric liftings . . . . .	74
4.3	Reduced liftings . . . . .	77
4.4	Properties of the characteristic function . . . . .	86
4.5	Applications to completely positive maps . . . . .	90
4.6	Appendix . . . . .	98
<b>5</b>	<b>Constrained Liftings</b>	<b>101</b>
5.1	Minimal constrained dilation . . . . .	103
5.2	Constrained characteristic function . . . . .	109
5.3	Appendix . . . . .	113
<b>6</b>	<b>Modules</b>	<b>117</b>
6.1	Invariants of Hilbert modules . . . . .	117
6.2	Examples . . . . .	122
6.3	Covariant representations, Hardy algebras . . . . .	125
6.4	Appendix . . . . .	137

# Chapter 1

## Introduction

This thesis specializes on certain topics in operator theory related to dilation theory and to the theory of invariant subspaces. For any contraction one associates an isometry (or a unitary) unique up to unitary equivalence called the minimal isometric (or unitary) dilation.

Earlier pioneering works in dilation theory were by Sz. Nagy, Stinespring and Kolmogorov. Since then it has been one of the central techniques for working with interpolation, extension and similarity problems, noncommutative probability, semigroups of completely positive maps and operator spaces. Surveys of such applications can be found in [FF90], [Ar69], [Pis01], [Pau03], [BP94], [Bh96] and [ER00].

Since late 90s there has been a lot of interest in dilation theory of row contractions ([Ar98],[DKS01],[Po89a],[Po02],[BJKW00]). A row contraction is a tuple of operators on a common Hilbert space which as a row operator is contractive. One instance when row contractions appear is during Kraus decomposition of normal unital completely positive maps. In place of the shift operator on  $\ell^2$  (or  $H^2$ ) appearing in the construction of dilation of a single operator, the shift operators on the Fock space is needed in the context of the dilation of a row contraction. Arveson used

this type of dilation to study some module structures induced by row contractions.

In this thesis we investigate normal unital completely positive maps on  $B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$  with invariant vector states and solve the classification problem of coisometric row contractions. There are many recent works on non-coisometric case of this classification problem and much of this case is well understood. We prove several results about fixed points of completely positive maps on  $B(\mathcal{H})$ . The role of theories of fixed points and ergodicity is significant for von Neumann algebras. A surprising observation of Arveson is that certain normal unital completely positive maps on  $B(\mathcal{H})$  produces geometric invariants. Our above mentioned classification results are related to these invariants. Finally we extend our theory to certain algebras on Hilbert modules which includes analytic cross-products as special case.

Using any state or rather any positive linear functional on a given  $C^*$ -algebra  $\mathcal{A}$ , we get an embedding of  $\mathcal{A}$  as a subalgebra of the algebra of bounded operators  $B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$  through the GNS construction. Unital completely positive maps are analogs of states when the codomain  $\mathbb{C}$  is replaced by a  $C^*$ -algebra. It was Stinespring who first realised that unital completely positive maps from  $C^*$ -algebras to any  $B(\mathcal{H})$  can be dilated to representations of those  $C^*$ -algebras. Completely positive maps have been used extensively in operator algebras and its classification, and in mathematical physics. So the problem of classifying completely positive maps is important.

Another significant result in the theory of invariant subspaces of interest to us here is Beurling's theorem. It states that any invariant subspace of  $H^2$  for the shift operator can be realised as the range of an inner  $H^\infty$  function. Sz.Nagy and Foias classify certain big classes of contractions (namely the completely noncounitary ones) up to unitary equivalence us-

ing some  $H^\infty$  functions called the characteristic functions which are identifiable with those from Beurling's theorem.

The notion of characteristic function and the related theory was extended to row contractions by Popescu using his theory of dilation of row contractions. He developed a noncommutative analogue of  $H^\infty$  for this purpose. Some analytic characteristic functions in the sense of multivariable complex analysis for commuting row contractions were studied in [BES05], [Po05], [BT07], etc. A function intrinsically similar to this analytic function already appeared in a preceding article ([Ar98]) of Arveson.

The Cuntz algebras are primary example of simple nuclear purely infinite  $C^*$ -algebras. Representations of Cuntz algebras are obtained from minimal isometric dilations of coisometric row contractions. These representations come very handy as a tool for investigating endomorphisms. The commutant lifting theorem of Bratteli, Jorgensen, et al. and several other of their works illustrates this. Further, the irreducible endomorphisms on  $B(\mathcal{H})$  with unique invariant vector states via Kraus decomposition yield minimal isometric dilations of certain tuples of complex numbers corresponding to Cuntz states.

The weak Markov dilation (cf. [BP94]) of an ergodic completely positive map (or discrete completely positive semigroup) on  $B(\mathcal{H})$  for some  $\mathcal{H}$  with an invariant vector state is always an irreducible endomorphism. While looking at the infinite tensor product picture of weak Markov dilations, certain convergence results (quoted as Appendix at the end of Chapter 3) had made us to look for (or conjecture the existence of) multianalytic operators similar to Sz. Nagy-Foias/Popescu's characteristic functions. We compute these extended characteristic functions using techniques from noncommutative probability in Section 3. They are complete unitary invariants for a class of completely positive maps on  $B(\mathcal{H})$ . This approach is motivated by the scattering theory for noncommutative

Markov chains of Kümmerer and Maassen.

In chapter 4 we reinterpret the above phenomenon of occurrence of multianalytic invariants and thereby we notice that it has to do with a much more general setup, namely that of subisometric liftings. Such liftings for single contractions appeared first in a work of Douglas and Foias [DF84]. They established related uniqueness and commutant lifting properties. In this context we introduce characteristic functions of contractive liftings of row contractions. They classify the certain liftings up to unitary equivalence and provide a kind of functional model. The most general setting is identified here which we call reduced liftings. Finally we discuss a factorization property of these multianalytic functions and provide applications of our theory to completely positive maps. Coisometric lifting is one of the most useful special case. It relates via Kraus decomposition to the lifting of the corresponding normal unital completely positive map. The commutant lifting theorem of Bratteli, Jorgensen, Kishimoto and Werner gives an affine order isomorphism between the fixed points of normal unital completely positive maps on some  $B(\mathcal{H})$  and the commutants of the Cuntz algebra representations coming from the associated Stinespring dilation. We apply our theory along with this lifting theorem to prove a one-to-one correspondence between the fixed point sets of any two normal unital completely positive maps on algebras of bounded operators on Hilbert spaces where one of the map is a coisometric lifting (or power dilation) of the other by a  $*$ -stable row contraction.

Apart from the minimal isometric dilations there are other types of dilations (see [Po05], [Ar98], [BBD04], [BB02] and [BDZ06]) called constrained dilations for row contractions satisfying polynomial relations. An important class of the constrained dilations is that of standard commuting dilation initiated in the works of Drury [Dr78] and Arveson [Ar98] and it comes with a multi-variable analytic functional calculus model. David-



son [DKS01] gave the complete structure of minimal isometric dilations for row contractions on finite dimensional spaces. Some of his techniques help us in chapter 5 to explore how constrained dilations can be derived from minimal isometric dilations in the context of row contractions on finite dimensional Hilbert space. Our above theory of multi-analytic operators also extends to constrained liftings.

A byproduct of the study of row contractions is the development of a noncommutative multivariable complex analysis by Popescu. Many results of classical theory like theorems of Cauchy, Liouville, Schwartz, etc. have their noncommutative analogs. Here the open unit disc in the complex plane is replaced by  $[B(\mathcal{H})^d]_1 := \{(X_1, \dots, X_d) \in B(\mathcal{H})^d : \|X_1X_1^* + \dots + X_dX_d^*\| < 1\}$ , for some Hilbert space  $\mathcal{H}$ .

There after we obtain that the two invariants of Hilbert modules, namely, curvature invariant and Euler characteristic are related to the characteristic function. In the last section of chapter 6 we take a coordinate free approach to our theory using Hilbert  $C^*$ -modules and their von Neumann counterparts. Hilbert modules first appeared in the works of Kaplansky while attempting to generalize Hilbert spaces by structures where the inner products can take values in commutative  $C^*$ -algebras. The motivation was to use the rich theory of vector bundles in operator algebras. Later Paschke and Rieffel independently extended this study to the case where the inner product take values in arbitrary  $C^*$ -algebras. Today they are extensively used in KK-theory, classification of  $C^*$ -algebras and noncommutative dynamics.

Cuntz-Pimsner algebras are quotients of certain subalgebras (namely, of Toeplitz algebras) of  $C^*$ -algebras of adjointable operators on Hilbert bimodules. They contain crossed products by  $\mathbb{Z}$  and Cuntz-Krieger algebras as special cases. The von Neumann counterpart of Toeplitz algebras are “Hardy algebras”. A notion of analytic crossed product can be given

using Hardy algebras. The Hardy algebras were shown by Muhly and Solel to be generalizations of  $H^\infty$  in a way similar to the noncommutative multivariable complex analysis of Popescu mentioned above.

We have attempted to present the beautiful mathematical theory which involves an interplay between the theory of invariant subspaces, multivariable (noncommutative) complex analysis, the theory of completely positive maps and dilation theory, and we discuss some of its implications in operator algebras. The related theory has developed significantly in recent years and we sincerely hope that our monograph will be an invitation for the readers to pursue this promising topic.

**Acknowledgements:** I want to thank Rolf Gohm with whom I have collaborated for two of the articles which are part of the present thesis and for permitting me to include them here. These two articles are chapter 3 and 4 here. Discussions with him has been very enriching and helped me to improve my mathematical skills. I am indebted to M. Schürmann for his encouraging me to work on this monograph. I acknowledge the significant amount of Mathematics I learned from the mathematical expertise of B.V.R. Bhat, M. Schürmann and U. Franz. I am grateful to G. Elliott, H. Osaka, J. Zacharias and many other with whom I had discussions on these mathematical topics. Much of the work in this monograph has been deeply influenced by the works of W. Arveson, G. Popescu, R. Gohm, P. Muhly and B. Solel. I am thankful to my colleagues at the Mathematics Department of the University of Greifswald where most of the work was down. Research work done during my stay at Fields Institute, Toronto in summer 2007 and at Ritsumeiken University Kyoto in summer 2008 has been very useful for this thesis, and I gratefully acknowledge the financial support I received for my visit at these Institutes.

# Chapter 2

## Dilations, Beurling's Theorem and Other Preliminaries

### 2.1 Minimal isometric dilations

Isometries and unitaries have several preferred properties as compared to contractions. In dilation theory one utilizes the existence of an isometric dilation of a given contraction. Consider the simplest case of a contraction, namely a scalar operator  $k$  on  $\mathbb{C}$  with  $|k| \leq 1$ . The “smallest” isometric dilation of  $k$  will be the operator  $\tilde{k} \in B(\mathbb{C} \oplus l^2(\mathbb{N}))$  given by

$$\tilde{k}(h, h_1, h_2, \dots) := (kh, (1 - |k|^2)^{\frac{1}{2}}h, h_1, h_2, \dots)$$

where  $h, h_i \in \mathbb{C}$  for  $i = 1, 2, \dots$  and  $\sum_{i=1}^d |h_i|^2 < \infty$ . The isometric dilation is on a rather big space as compared to the domain of the initial operator.

It is known that every contraction has isometric dilations and uniqueness up to unitary equivalence of isometric dilations can be ensured by making minimality assumption. Formally, we have:

**Definition 2.1.1.** *Let  $T$  be a contraction on a Hilbert space  $\mathcal{H}$ , i.e.,  $\|T\| \leq$*

1. An isometry  $V$  on Hilbert space  $\tilde{\mathcal{H}}$  is an isometric dilation of  $T$  if

$$\mathcal{H} \subset \tilde{\mathcal{H}} \quad \text{and} \quad P_{\mathcal{H}} V^n|_{\mathcal{H}} = T^n.$$

An isometric dilation is said to be minimal if  $\tilde{\mathcal{H}} = \overline{\text{span}}\{V^n h : n \in \mathbb{N} \cup \{0\}, h \in \mathcal{H}\}$ .

A minimal isometric dilation (mid for short) can be constructed in a parallel way to the isometry constructed in the example above.

A tuple  $\underline{T} = (T_1, \dots, T_d)$  of bounded operators on a Hilbert space  $\mathcal{H}$  is said to be a row contraction if  $T_1 T_1^* + \dots + T_d T_d^* \leq \mathbf{1}$ . Treating a row contraction  $\underline{T}$  as a row operator from  $\bigoplus_{i=1}^d \mathcal{H}$  to  $\mathcal{H}$ , define  $D_* := (\mathbf{1} - \underline{T} \underline{T}^*)^{\frac{1}{2}} : \mathcal{H} \rightarrow \mathcal{H}$  and  $D := (\mathbf{1} - \underline{T}^* \underline{T})^{\frac{1}{2}} : \bigoplus_{i=1}^d \mathcal{H} \rightarrow \bigoplus_{i=1}^d \mathcal{H}$ . This implies that

$$D_* = (\mathbf{1} - \sum_{i=1}^d T_i T_i^*)^{\frac{1}{2}}, \quad D = (\delta_{ij} \mathbf{1} - T_i^* T_j)^{\frac{1}{2}}_{d \times d}. \quad (2.1)$$

Observe that  $\underline{T} D^2 = D_*^2 \underline{T}$  and hence  $\underline{T} D = D_* \underline{T}$ . Let  $\mathcal{D} := \text{Range } D$  and  $\mathcal{D}_* := \text{Range } D_*$ .

We use the following *multi-index notation*. Let  $\Lambda$  denote the set  $\{1, 2, \dots, d\}$  and  $\tilde{\Lambda} := \bigcup_{n=0}^{\infty} \Lambda^n$ , where  $\Lambda^0 := \{0\}$ . If  $\alpha \in \Lambda^n \subset \tilde{\Lambda}$  the integer  $n = |\alpha|$  is called its length. Now  $T_{\alpha}$  with  $\alpha = (\alpha_1, \dots, \alpha_n) \in \Lambda^n$  means  $T_{\alpha_1} T_{\alpha_2} \dots T_{\alpha_n}$ .

The full Fock space over  $\mathbb{C}^d$  ( $d \geq 2$ ) denoted by  $\Gamma(\mathbb{C}^d)$  is the Hilbert space

$$\Gamma(\mathbb{C}^d) = \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes 2} \oplus \dots \oplus (\mathbb{C}^d)^{\otimes m} \oplus \dots$$

We will just write  $\Gamma$  instead of  $\Gamma(\mathbb{C}^d)$  at times. The element  $1 \oplus 0 \oplus \dots$  of  $\Gamma(\mathbb{C}^d)$  is called the vacuum vector. Let  $\{e_1, \dots, e_d\}$  be the standard orthonormal basis of  $\mathbb{C}^d$ . We include  $d = \infty$  in which case  $\mathbb{C}^d$  stands for a

complex separable Hilbert space of infinite dimension. For  $\alpha \in \tilde{\Lambda}$ ,  $e_\alpha$  will denote the vector  $e_{\alpha_1} \otimes e_{\alpha_2} \otimes \dots \otimes e_{\alpha_m}$  in the full Fock space  $\Gamma(\mathbb{C}^d)$  and  $e_0$  will denote the vacuum vector. Then the (left) creation operators  $L_i$ 's on  $\Gamma(\mathbb{C}^d)$  are defined by

$$L_i x = e_i \otimes x$$

for  $1 \leq i \leq d$  and  $x \in \Gamma(\mathbb{C}^d)$ . The row contraction  $\underline{L} = (L_1, \dots, L_d)$  consists of isometries with orthogonal ranges.

Popescu in [Po89a] gave the following explicit presentation of the minimal isometric dilation of  $\underline{T}$  by  $\underline{V}$  on  $\hat{\mathcal{H}} = \mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D})$ ,

$$V_i(h \oplus \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha) = T_i h \oplus [e_0 \otimes D_i h + e_i \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha] \quad (2.2)$$

for  $h \in \mathcal{H}$  and  $d_\alpha \in \mathcal{D}$ . Here  $D_i h := D(0, \dots, 0, h, 0, \dots, 0)$  and  $h$  is embedded at the  $i^{\text{th}}$  component. For  $d = 1$  this coincides with Schäffer's construction.

In other words, the  $V_i$  are isometries with orthogonal ranges such that  $T_i^* = V_i^*|_{\mathcal{H}}$  for  $i = 1, \dots, d$  and the spaces  $V_\alpha \mathcal{H}$  with  $\alpha \in \tilde{\Lambda}$  together span the Hilbert space on which the  $V_i$  are defined. It is an important fact, which we shall use repeatedly, that such minimal isometric dilations are unique up to unitary equivalence (cf. [Po89a]). If in addition  $\sum T_i T_i^* = \mathbf{1}$ , then  $\sum V_i V_i^* = \mathbf{1}$ .

For  $d \geq 2$ , the Cuntz algebra  $\mathcal{O}_d$  is the  $C^*$ -algebra generated by  $n$ -isometries  $\underline{s} = \{s_1, \dots, s_d\}$ , satisfying Cuntz relations:  $s_i^* s_j = \delta_{ij} \mathbf{1}$ ,  $1 \leq i, j \leq n$ , and  $\sum s_i s_i^* = \mathbf{1}$ . Therefore,  $V_i$ 's are representations of Cuntz algebras, if  $\sum T_i T_i^* = \mathbf{1}$ .

## 2.2 Kraus decomposition, and theorems of Stinespring and Beurling

Any normal unital completely positive map on  $B(\mathcal{H})$  for any Hilbert space  $\mathcal{H}$  is associated to a coisometric row contraction (may be of infinite length) on  $\mathcal{H}$ .

**Theorem 2.2.1.** (*Kraus*) *Let  $\varphi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  be a normal unital completely positive map. Then there is a row contraction  $\underline{T} = (T_1, \dots, T_d)$  such that*

$$\varphi(X) = \sum_{i=1}^d T_i X T_i^*, \quad \text{for } X \in B(\mathcal{H}),$$

*where  $d$  may be infinite. In the  $d = \infty$  case the convergence of the above sum is to be taken in strong operator topology.*

The unital completely positive maps have proved to be the right generalisations of states in the study of operator algebras. A role parallel to the GNS construction is played by the following theorem of Stinespring for completely positive maps:

**Theorem 2.2.2.** *If  $\varphi$  is a unital completely positive map from a unital  $C^*$ -algebra  $\mathcal{A}$  to into  $B(\mathcal{H})$ , then there is a Hilbert space  $\mathcal{K}$  containing  $\mathcal{H}$  and  $*$ -representation  $\sigma$  of  $\mathcal{A}$  on  $\mathcal{K}$  such that  $\varphi(X) = P_{\mathcal{H}}\sigma(X)|_{\mathcal{H}}$ .*

An elegant way of deriving Kraus decomposition from Stinespring Theorem is described in ([Go04], page 48).

Another result in operator theory crucial for us (cf. Section 4.1) is the Beurling type theorem proved by Popescu (cf. [Po89b]). We give here a detailed proof as there is only a short discussion on it in the existing

literature. At first we quote the corresponding classical theorem. Let  $\mathbb{D}$  denote the open unit ball of  $\mathbb{C}$ . For a Hilbert space  $\mathcal{K}$  we denote the set

$$\{u \mid u(z) = \sum_0^k z^k a_k \text{ for } z \in \mathbb{D}, a_k \in \mathcal{K} \text{ and } \sup_{0 < r < 1} \int_0^{2\pi} \|u(re^{it})\|^2 dt < \infty\}.$$

by  $H^2(\mathcal{K})$  and define

$$H^\infty := \{u : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}} \mid u \text{ is analytic on } \mathbb{D}, \text{ a.e. continuous on } \partial\mathbb{D}, \\ \text{and } \sup_{z \in \mathbb{D}} |u(z)| < \infty\}.$$

Moreover, a function  $u \in H^\infty$  is called *inner* if  $|u(e^{it})| = 1$  a.e. for  $t \in [0, 2\pi)$ .

**Theorem 2.2.3.** (*Beurling's Theorem*) *The invariant subspaces  $\mathcal{M}$  for the multiplication operator  $M_z$  on  $H^2(\mathcal{U})$ , for some Hilbert space  $\mathcal{U}$ , are precisely those of the form*

$$\mathcal{M} = \theta H^2(\mathcal{N})$$

where  $\theta$  is an inner function with values in  $B(\mathcal{N}, \mathcal{U})$ .

Let  $\underline{V} = (V_1, \dots, V_d)$  be a row contraction consisting of isometries on a Hilbert space  $\mathcal{R}$ . It follows that  $V_i, i = 1, \dots, d$  have mutually orthogonal ranges. A subspace  $\mathcal{K}$  of  $\mathcal{R}$  is called *wandering* if

$$V_\alpha \mathcal{K} \perp V_\beta \mathcal{K} \quad \text{for distinct } \alpha, \beta \in \tilde{\Lambda}.$$

Further, a row contraction  $\underline{V}$  of isometries is called a *d-orthogonal shift* if there is a wandering subspace  $\mathcal{K} \subset \mathcal{R}$ , and

$$\mathcal{R} = M(\mathcal{K}) := \overline{\text{span}}\{V_\alpha \mathcal{K} : \alpha \in \tilde{\Lambda}\}.$$

We have a unitary transformation  $\Phi^\mathcal{K} : M(\mathcal{K}) \rightarrow \Gamma \otimes \mathcal{K}$  defined by

$$\Phi^\mathcal{K} \left( \sum_{\alpha \in \tilde{\Lambda}} V_\alpha k_\alpha \right) = \sum_{\alpha \in \tilde{\Lambda}} (L_\alpha \otimes \mathbf{1})(e_0 \otimes k_\alpha).$$

**Theorem 2.2.4.** (cf. [Po89a]) (Wold decomposition) Let  $\underline{V} = (V_1, \dots, V_d)$  be a row contraction consisting of isometries on a common Hilbert space  $\mathcal{R}$ . Then  $\mathcal{R}$  decomposes as  $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$ , such that  $\mathcal{R}_0$  and  $\mathcal{R}_1$  reduces  $V_i, i = 1, \dots, d$ , and we have

(a)  $(\mathbf{1} - \sum_{i=1}^d V_i V_i^*)|_{\mathcal{R}_1} = 0$  and  $(V_1|_{\mathcal{R}_0}, \dots, V_d|_{\mathcal{R}_0})$  is a  $d$ -orthogonal shift for  $\mathcal{R}_0$ .

(b)  $\mathcal{R}_1 = \cap_{n=0}^{\infty} \overline{\text{span}}\{V_{\alpha}\mathcal{R} : |\alpha| = n\}$  and  $\mathcal{R}_0 = M(\mathcal{N})$ , where  $\mathcal{N} := \mathcal{R} \ominus \overline{\text{span}}\{V_i\mathcal{R} : i = 1, \dots, d\}$ .

For two Hilbert spaces  $\mathcal{K}$  and  $\mathcal{K}'$  consider an operator  $\theta : \mathcal{K} \rightarrow \Gamma \otimes \mathcal{K}'$ . Take  $M_{\theta} : \Gamma \otimes \mathcal{K} \rightarrow \Gamma \otimes \mathcal{K}'$  as the operator:

$$M_{\theta}(L_{\alpha} \otimes \mathbf{1})(e_0 \otimes k) := (L_{\alpha} \otimes \mathbf{1})\theta k, \quad k \in \mathcal{K}.$$

That  $M_{\theta}(L_i \otimes \mathbf{1}) = (L_i \otimes \mathbf{1})M_{\theta}$  is immediate. The operator  $\theta$  is called *inner*, if  $M_{\theta}$  is an isometry.

**Lemma 2.2.5.** Let  $\underline{V}$  and  $\underline{V}'$  be two  $d$ -orthogonal shifts on Hilbert spaces  $\mathcal{R}$  and  $\mathcal{R}'$ , with the wandering subspaces  $\mathcal{K}$  and  $\mathcal{K}'$  respectively. Let  $Q$  be a contraction of  $\mathcal{R}$  into  $\mathcal{R}'$  such that for  $i = 1, \dots, d$

$$QV_i = V'_i Q.$$

Then there is a contraction  $\theta$  of  $\mathcal{K}$  into  $\Gamma \otimes \mathcal{K}'$  such that

$$\Phi^{\mathcal{K}'} Q = M_{\theta} \Phi^{\mathcal{K}}. \quad (2.3)$$

If  $Q$  is an isometry, then  $\theta$  is inner.

*Proof.* Let  $k$  be an arbitrary element in  $\mathcal{K}$ . Then from the definition of  $Q$  we get elements  $k'_{\alpha} \in \mathcal{K}'$ , such that

$$Qk = \sum_{\alpha \in \tilde{\Lambda}} V'_{\alpha} k'_{\alpha}.$$



It yields  $\sum \|k'_\alpha\|^2 = \|Qk\|^2 \leq \|k\|^2$ . We set  $k'_\alpha = \theta_\alpha k$  for all  $k \in \mathcal{K}$  and thereby get  $\theta_\alpha \in B(\mathcal{K}, \mathcal{K}')$ . The function

$$\theta k := \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes \theta_\alpha k, \quad k \in \mathcal{K}$$

is contractive. Now Equation (2.3) can be established in the following way:

We first observe that

$$\Phi^{\mathcal{K}'} Qk = \theta k, \quad k \in \mathcal{K}.$$

Therefore for  $k_\alpha \in \mathcal{K}$

$$\begin{aligned} \Phi^{\mathcal{K}'} Q \sum_{\alpha \in \tilde{\Lambda}} V_\alpha k_\alpha &= \sum_{\alpha \in \tilde{\Lambda}} \Phi^{\mathcal{K}'} V'_\alpha Qk_\alpha = \sum_{\alpha \in \tilde{\Lambda}} (L_\alpha \otimes \mathbf{1}) \Phi^{\mathcal{K}'} Qk_\alpha \\ &= \sum_{\alpha \in \tilde{\Lambda}} (L_\alpha \otimes \mathbf{1}) \theta k_\alpha = M_\theta \sum_{\alpha \in \tilde{\Lambda}} (L_\alpha \otimes \mathbf{1}) (e_0 \otimes k_\alpha) \\ &= M_\theta \Phi^{\mathcal{K}} \sum_{\alpha \in \tilde{\Lambda}} V_\alpha k_\alpha. \end{aligned}$$

As  $\Phi^{\mathcal{K}}$  and  $\Phi^{\mathcal{K}'}$  are unitaries, we will have  $M_\theta$  to be an isometry, if  $Q$  is an isometry.  $\square$

**Theorem 2.2.6.** (*Beurling type theorem*) *Let  $\mathcal{U}$  be a Hilbert space. A subspace  $\mathcal{M}$  of  $\Gamma \otimes \mathcal{U}$  is invariant for  $L_i \otimes \mathbf{1}, i = 1, \dots, d$  if and only if*

$$\mathcal{M} = M_\theta(\Gamma \otimes \mathcal{N})$$

*for some Hilbert space  $\mathcal{N}$  and an inner function  $\theta : \mathcal{N} \rightarrow \Gamma \otimes \mathcal{U}$ .*

*Proof.* The proof of the “necessary” condition is trivial. We prove the “sufficient” condition. We consider the embedding of  $\mathcal{U}$  in  $\Gamma \otimes \mathcal{U}$  as  $e_0 \otimes \mathcal{U}$ , where  $e_0$  is the vacuum vector in the Fock space  $\Gamma$ . We set

$V_i = (L_i \otimes \mathbf{1})|_{\mathcal{M}}, i = 1, \dots, d$ . The row contraction  $\underline{V} = (V_1, \dots, V_d)$  consists of isometries with orthogonal ranges and

$$\cap_{n=0}^{\infty} \overline{\text{span}}\{V_{\alpha}\mathcal{M} : |\alpha| = n\} = \cap_{n=0}^{\infty} \overline{\text{span}}\{(L_{\alpha} \otimes \mathbf{1})(\Gamma \otimes \mathcal{U}) : |\alpha| = n\} = \{0\}.$$

The Wold decomposition gives us

$$\mathcal{M} = \overline{\text{span}}\{V_{\alpha}\mathcal{N} : \alpha \in \tilde{\Lambda}\} = M(\mathcal{N}) \quad (2.4)$$

where  $\mathcal{N} := \mathcal{M} \ominus \overline{\text{span}}\{V_i\mathcal{M} : i = 1, \dots, d\}$ . We use Lemma 2.2.5 with

$$\begin{aligned} \mathcal{R} &= \mathcal{M}, & \mathcal{K} &= \mathcal{N}, \\ \mathcal{R}' &= \Gamma \otimes \mathcal{U}, & V'_i &= L_i \otimes \mathbf{1}, & \mathcal{K}' &= \mathcal{U} \end{aligned}$$

and  $Q = id : \mathcal{M} \rightarrow \Gamma \otimes \mathcal{U}$ . Thereby an inner function  $\theta : \mathcal{N} \rightarrow \Gamma \otimes \mathcal{U}$  is obtained with

$$\Phi^{\mathcal{U}}Q = M_{\theta}\Phi^{\mathcal{K}}.$$

The observation that  $\Phi^{\mathcal{U}}$  is identity leads us to

$$h = M_{\theta}\Phi^{\mathcal{K}}h \text{ for } h \in \mathcal{M}.$$

Finally this implies

$$\mathcal{M} = \theta\Phi^{\mathcal{K}}\mathcal{M} = M_{\theta}(\Gamma \otimes \mathcal{N}).$$

□

## 2.3 Characteristic function of Popescu

After the preparation done in the previous section we look at how Popescu's characteristic functions, which are unitary invariants for certain row contractions, are developed using Beurling type theorem. The other characteristic functions we introduce in later chapters bear many common features with the one discussed here.

Recall that for any contraction  $T$  on a Hilbert space  $\mathcal{H}$  the mid has the form

$$V = \begin{pmatrix} T & 0 \\ * & A \end{pmatrix}$$

acting on the dilation space  $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}_A$  for some Hilbert space  $\mathcal{H}_A$  on which  $A$  is defined. Here  $A$  is  $*$ -stable, i.e.,  $\lim_{n \rightarrow \infty} \|(A^*)^n h\| = 0$  for all  $h \in \mathcal{H}_A$ . Infact the defect space  $\mathcal{D}$  (or  $\mathcal{D}_T$ ) is a wandering subspace for  $\mathcal{H}_A$  with respect to  $A$ .

At first consider the easy case, namely, of assuming  $T$  to be  $*$ -stable. Then the shift on  $l^2 \otimes \mathcal{D}_*$  is a mid of  $T$ . Schäffer's construction (or Popescu's construction for  $d = 1$  case) described before gives another realisation of the mid of  $T$ . Here we observe that  $\mathcal{H}_A$  is embedded as an invariant subspace for this shift. By Beurling's theorem an inner, operator valued  $H^\infty$  function  $\theta$  exists such that

$$\mathcal{H}_A = \theta(l^2 \otimes \mathcal{H}')$$

for some Hilbert space  $\mathcal{H}'$ . (To be precise, we need to make an identification of the shift on  $l^2$  with the one on  $H^2$ .)

A specific choice of such  $\theta$  is the *characteristic function*  $\theta_T$  for  $T$ . It is explicitly defined as

$$\theta_T(z) := (-T + zD_*(\mathbf{1} - zT^*)^{-1}D)|_{\mathcal{D}} \quad (2.5)$$

where  $z$  belongs to the open unit ball of the complex plane. Here  $\theta$  is bounded analytic function on the open unit ball taking values in  $B(\mathcal{D}, \mathcal{D}_*)$  and

$$\|\theta_T(z)\| = 1 \text{ a.e.} \quad \text{for } \|z\| = 1.$$

Obviously, we now have  $\mathcal{H}_A = \theta_T(l^2 \otimes \mathcal{D})$ . The identity (2.5) can be

expanded in power series as

$$\theta_T(z) := (-T + \sum_{n=1}^{\infty} z^n D_*(T^*)^{n-1} D)|_{\mathcal{D}}.$$

**Definition 2.3.1.** A row contraction  $\underline{T} = (T_1, \dots, T_d)$  is called

1. *\*-stable* if  $\lim_{n \rightarrow \infty} \sum_{|\alpha|=n} \|T_{\alpha}^* h\|^2 = 0$ ,
2. *completely non-coisometric (c.n.c. in short)* if

$$\{h \in \mathcal{H} : \sum_{|\alpha|=n} \|T_{\alpha}^* h\|^2 = \|h\|^2 \text{ for all } n \in \mathbb{N}\} = \{0\}.$$

In operator matrix form Popescu's construction of mid on  $\mathcal{H} \oplus (\Gamma \otimes \mathcal{D})$  is

$$V_i = \begin{pmatrix} T_i & 0 \\ D(e_i \otimes \cdot) & L_i \otimes \mathbf{1} \end{pmatrix}$$

In case of an arbitrary row contraction  $\underline{T}$ , one way of representing the mid is Popescu's construction. If  $\underline{T}$  is \*-stable, then the operator tuple  $\underline{L} \otimes \mathbf{1}$  on  $\Gamma \otimes \mathcal{D}$  is also a mid of  $\underline{T}$  and due to uniqueness of mids, there exist a unitary

$$W : \mathcal{H} \oplus (\Gamma \otimes \mathcal{D}) \rightarrow \Gamma \otimes \mathcal{D}_*$$

that intertwines between them. It is immediate that this  $W|_{\Gamma \otimes \mathcal{D}}$  is a multi-analytic inner operator, say  $M_{\theta} : \Gamma \otimes \mathcal{D} \rightarrow \Gamma \otimes \mathcal{D}_*$ . This is called the *characteristic function* of  $\underline{T}$  with symbol

$$\theta_T := M_{\theta}|_{e_0 \otimes \mathcal{D}} \text{ where } \theta_T : \mathcal{D} \rightarrow \Gamma \otimes \mathcal{D}_*.$$

Popescu introduced the above defined characteristic functions in [Po89b] and showed that for c.n.c. row contractions  $\underline{T}$  they are complete invariant

up to unitary equivalence. With  $P_j$  the orthogonal projection onto the  $j$ -th component,  $\theta_T$  in expanded form is

$$\theta_T d = -e_0 \otimes \sum_{j=1}^d T_j P_j d + \sum_{j=1}^d e_j \otimes \sum_{\alpha \in \bar{\Lambda}} e_\alpha \otimes D_* T_\alpha^* P_j D d, \quad d \in \mathcal{D}.$$

Bhattacharyya, Eschmeier and Sarkar defined in [BES05] an analytic characteristic function for any commuting row contraction  $\underline{T}$  as

$$\theta_T(z) := (-T + zD_*(\mathbf{1} - zT^*)^{-1}D)|_{\mathcal{D}}$$

where  $z$  belongs to the open unit ball of the  $\mathbb{C}^d$ . These functions are also complete invariants for c.n.c. operator tuples.

Take  $C : \mathcal{H} \rightarrow \Gamma \otimes \mathcal{D}_*$  to be the map

$$h \mapsto \sum_{\alpha \in \bar{\Lambda}} e_\alpha \otimes D_* T_\alpha^* h, \quad (2.6)$$

which is called the *Poisson kernel*. As  $\mathcal{D} \subset \bigoplus_{i=1}^d \mathcal{H}$  we take  $P_i$  as the projection of  $\mathcal{D}$  onto the  $i^{\text{th}}$  component,  $1 \leq i \leq d$ . From [Po89b] we know that the mid  $\underline{V}$  of a  $*$ -stable row contraction  $\underline{T}$  is unitarily equivalent to  $\underline{L} \otimes \mathbf{1}$ . In the following, using an argument similar to [FF90], IX.6.4, this unitary is constructed explicitly.

**Proposition 2.3.2.** *For a  $*$ -stable row contraction  $\underline{T}$  there exists a unitary*

$$\hat{W} : \mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}) \rightarrow \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_*$$

such that  $(L_i \otimes \mathbf{1})\hat{W} = \hat{W}V_i$ . Together with this intertwining relation it is determined by

$$\hat{W}|_{\mathcal{H}} = C, \quad \hat{W}|_{e_0 \otimes \mathcal{D}} = \theta_T.$$

*Proof.* It is easy to see that when  $\underline{T}$  is  $*$ -stable,  $C$  is an isometry (see [Po89b]) and  $(L_i^* \otimes \mathbf{1})C = CT_i^*$ . Let  $\mathcal{H}_* := C\mathcal{H}$  and  $T_{i*}^* := (L_i^* \otimes \mathbf{1})|_{\mathcal{H}_*}$ .

So  $\underline{L} \otimes \mathbf{1}$  is the mid of  $\underline{T}_*$ . Moreover  $\underline{T}_*$  is unitarily equivalent to  $\underline{T}$  as  $T_{i*}C_* = C_*T_i$  where  $C_*$  is the unitary given by  $C$  as a map from  $\mathcal{H}$  to  $\mathcal{H}_*$ .

Because mids are unique up to unitary equivalence (cf. [Po89a]) this implies that there is a unitary  $\hat{W} : \mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}) \rightarrow \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_*$  such that  $(L_i \otimes \mathbf{1})\hat{W} = \hat{W}V_i$  and  $\hat{W}|_{\mathcal{H}} = C$ . For  $h_i \in \mathcal{H}$  we have

$$\begin{aligned}
& \hat{W}(e_0 \otimes D(h_1, \dots, h_d)) = \hat{W} \sum_i (V_i - T_i)h_i \\
&= \sum_i (L_i \otimes \mathbf{1})Ch_i - \sum_j CT_jh_j \\
&= \sum_i e_i \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes D_*T_\alpha^*h_i - \sum_j \sum_{\beta \in \tilde{\Lambda}} e_\beta \otimes D_*T_\beta^*T_jh_j \\
&= -e_0 \otimes \sum_i D_*T_ih_i + \sum_j e_j \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes D_*T_\alpha^*P_jD^2(h_1, \dots, h_d) \\
&= -e_0 \otimes \sum_i T_iP_iD(h_1, \dots, h_d) \\
&+ \sum_j e_j \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes D_*T_\alpha^*P_jD(D(h_1, \dots, h_d)) \\
&= \theta_TD(h_1, \dots, h_d).
\end{aligned}$$

Hence  $\hat{W}|_{e_0 \otimes \mathcal{D}} = \theta_T$ . From the explicit form of Popescu's dilation we see that the restriction of  $V_i$  to  $\Gamma(\mathbb{C}^d) \otimes \mathcal{D}$  coincides with  $L_i \otimes \mathbf{1}$ . This shows that  $\hat{W}$  is determined by  $C$  and  $\theta_T$  (together with  $(L_i \otimes \mathbf{1})\hat{W} = \hat{W}V_i$ ).  $\square$

It is observed from this that  $\theta_T$  is an isometry when  $\underline{T}$  is  $*$ -stable (also realized in Remark 3.2 of [Po89b]).

The noncommutative Beurling type theorem (Theorem 2.2.6) and the characteristic functions of row contractions indicate that there should be a parallel theory of noncommutative complex analysis. Such a theory was given by Popescu ([Po91],[Po06]). There one replaces open unit disc of the complex plane by  $[B(\mathcal{H})^d]_1 := \{(X_1, \dots, X_d) \in B(\mathcal{H})^d : \|X_1X_1^* + \dots +$

$X_d X_d^* \| < 1\}$ . Let  $Z_1, \dots, Z_d$  denote  $d$  noncommuting variables and

$$Hol(\mathbb{D}_d^N) := \{F = \sum_{\alpha \in \bar{\Lambda}} a_\alpha Z_\alpha : \limsup_{k \rightarrow \infty} (\sum_{|\alpha|=k} |a_\alpha|^2)^{\frac{1}{2k}} \leq 1\}.$$

Then the noncommutative analogue of  $H^\infty$  is given by

$$H^\infty(\mathbb{D}_d^N) = \{F \in Hol(\mathbb{D}_d^N) : \|F\|_\infty := \sup \|F(X_1, \dots, X_d)\| < \infty$$

where supremum is taken over  $(X_1, \dots, X_d)$  in  $[B(\mathcal{H})^d]_1\}$ ,

Note that  $\mathbb{D}_d^N$  alone has no explicit meaning and the notation  $Hol(\mathbb{D}_d^N)$  is used to denote the set which is the counterpart of the set of classical analytic functions on the open unit ball of the complex plane.

## 2.4 Hilbert $C^*$ -module

Next we review some selected topics about Hilbert  $C^*$ -modules which we need in Section 6.3.

For a given  $C^*$ -algebra  $\mathcal{A}$ , a pre-Hilbert  $C^*$ -module  $\mathcal{G}$  on  $\mathcal{A}$  is a right  $\mathcal{A}$ -module with a sesquilinear inner product:

$$\langle \cdot, \cdot \rangle : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{A},$$

for  $\chi, \eta \in \mathcal{G}, a \in \mathcal{A}$  such that:

$$(a) \quad \langle \chi, \eta a \rangle = \langle \chi, \eta \rangle a,$$

$$(b) \quad \langle \chi, \chi \rangle \geq 0 \text{ and}$$

$$(c) \quad \langle \chi, \chi \rangle = 0 \Leftrightarrow \chi = 0$$

A norm can be defined on  $\mathcal{G}$  by  $\|\cdot\| := \|\langle \cdot, \cdot \rangle\|_{\mathcal{A}}^{\frac{1}{2}}$  where  $\|\cdot\|_{\mathcal{A}}$  is the  $C^*$ -algebra norm of  $\mathcal{A}$ . Such a module  $\mathcal{G}$  is called *Hilbert  $C^*$ -module* on  $\mathcal{A}$  if

it is complete with respect to the above norm. By  $\mathcal{L}(\mathcal{G})$  we denote the set of all adjointable maps on  $\mathcal{G}$ . The set  $\mathcal{L}(\mathcal{G})$  turns out to be a  $C^*$ -algebra with respect to the operator norm induced by  $(\mathcal{G}, \|\cdot\|)$ .

Let  $\varphi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{G})$  be a  $*$ -homomorphism and write  $\varphi(a)\eta := a\eta$  for all  $a \in \mathcal{A}, \eta \in \mathcal{G}$ . The Hilbert  $C^*$ -module  $\mathcal{G}$  together with such a  $*$ -homomorphism is called a *Hilbert bimodule* on  $\mathcal{A}$ . There is an associated tensor product structure  $\mathcal{G} \otimes \mathcal{G}$  for any Hilbert bimodule  $\mathcal{G}$  given by

$$\langle \eta_1 \otimes \chi_1, \eta_2 \otimes \chi_2 \rangle = \langle \chi_1, \langle \eta_1, \eta_2 \rangle \chi_2 \rangle \quad \text{for } \eta_1, \eta_2, \chi_1, \chi_2 \in \mathcal{G}.$$

**Definition 2.4.1.** A  $W^*$ -correspondence  $\mathcal{E}$  over a von Neumann algebra  $\mathcal{M}$  is a self-dual Hilbert  $\mathcal{M}$ -bimodule where the corresponding left action  $\varphi$  of  $\mathcal{M}$  on  $\mathcal{E}$  is normal.

**Definition 2.4.2.** A pair  $(T, \sigma)$  is called a covariant representation of  $W^*$ -correspondence  $\mathcal{E}$  over  $\mathcal{M}$  on a Hilbert space  $\mathcal{H}$ , if:

- (a)  $T : \mathcal{E} \rightarrow B(\mathcal{H})$  is a linear map that is continuous w.r.t. the  $\sigma$ -topology of [BDH88] on  $\mathcal{E}$  and the ultraweak topology on  $B(\mathcal{H})$ .
- (b)  $\sigma : \mathcal{M} \rightarrow B(\mathcal{H})$  is a normal homomorphism.
- (c)  $T(a\xi) = \sigma(a)T(\xi), T(\xi a) = T(\xi)\sigma(a)$  for all  $\xi \in \mathcal{E}, a \in \mathcal{M}$ .

If a completely contractive covariant representation satisfies

$$T(\xi)^*T(\eta) = \sigma(\langle \xi, \eta \rangle)$$

in addition, it is called *isometric*.

One of our prime objective is to use dilation theory of covariant representations of  $W^*$ -correspondences to extend our theory of characteristic functions and then apply it to classify these covariant representations. First we remark that for any  $W^*$ -correspondence  $\mathcal{E}$  over a von Neumann



algebra  $\mathcal{M}$  and a normal representation  $\sigma : \mathcal{M} \rightarrow B(\mathcal{H})$  there is an induced tensor product  $\mathcal{E} \otimes_\sigma \mathcal{H}$ , which is a Hilbert space, with the defining identities:

$$\begin{aligned}\xi_1 a \otimes \eta_1 &= \xi_1 \otimes \sigma(a) \eta_1 \quad \text{and} \\ \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle &= \langle \eta_1, \sigma(\langle \xi_1, \xi_2 \rangle) \eta_2 \rangle\end{aligned}$$

for  $\xi_1, \xi_2 \in \mathcal{E}$  and  $\eta_1, \eta_2 \in \mathcal{H}$ . In addition if  $\sigma$  is also faithful, define

$$\mathcal{E}^\sigma := \{\mu \in B(\mathcal{H}, \mathcal{E} \otimes_\sigma \mathcal{H}) : \mu \sigma(a) = (\varphi(a) \otimes \mathbf{1}) \mu \quad \forall a \in \mathcal{M}\}.$$

$\mathcal{E}^\sigma$  is called the  $\sigma$ -dual of  $\mathcal{E}$ . Its open unit ball is denoted by  $\mathbb{D}(\mathcal{E}^\sigma)$ .

For every covariant representation  $(T, \sigma)$  on  $\mathcal{H}$ , we can get an operator  $\tilde{T}$  in  $B(\mathcal{E} \otimes \mathcal{H}, \mathcal{H})$  given by

$$\tilde{T}(\eta \otimes h) := T(\eta)h.$$

It is handy to work with the operator  $\tilde{T}$  instead of  $(T, \sigma)$  in dilation theory. The next lemma provides a dictionary to translate some important notions for  $(T, \sigma)$  in terms of that for  $\tilde{T}$  and vice versa.

**Lemma 2.4.3.** *[MS98]*

(a)  *$T$  is completely bounded if and only if  $\tilde{T}$  is bounded and*

$$\|T\|_{cb} = \|\tilde{T}\|$$

(b)  *$T$  is completely contractive  $\Leftrightarrow \|\tilde{T}\| \leq 1$ .*

(c)  *$(T, \sigma)$  is isometric if and only if  $\tilde{T}$  is isometric.*

*Proof.* We prove only the part (b) here. Start with the assumption that  $\|\tilde{T}\| \leq 1$ . Let  $(\eta_{ij})_{k \times k}$  be an element in  $M_k(\mathcal{E})$ . For  $h = (h_1, \dots, h_k), h_i \in \mathcal{H}$

we have

$$\begin{aligned}
\|(T(\eta_{ij}))_{k \times k} h\|^2 &= \sum_{i=1}^k \left\| \sum_{j=1}^k T(\eta_{ij}) h_j \right\|^2 = \sum_{i=1}^k \left\| \tilde{T} \left( \sum_{j=1}^k \eta_{ij} \otimes h_j \right) \right\|^2 \\
&\leq \sum_{i=1}^k \left\| \left( \sum_{j=1}^k \eta_{ij} \otimes h_j \right) \right\|^2 = \sum_{i,j,l} \langle \eta_{ij} \otimes h_j, \eta_{il} \otimes h_l \rangle \\
&= \sum_{i,j,l} \langle h_j, \sigma(\langle \eta_{ij}, \eta_{il} \rangle) h_l \rangle.
\end{aligned}$$

Therefore

$$\|(T(\eta_{ij}))_{k \times k} h\|^2 \leq \left\| \left( \sum_{i=1}^k \sigma(\langle \eta_{ij}, \eta_{il} \rangle) \right)_{k \times k} \right\| \|h\|^2.$$

$\sigma$  is completely contractive because it is a  $*$ -representation. This implies

$$\|(T(\eta_{ij}))_{k \times k} h\|^2 \leq \|(\eta_{ij})_{k \times k}\|^2 \|h\|^2,$$

Hence  $\|T\|_{cb} \leq 1$ .

For the converse first we claim that: If  $T$  is completely contractive, then for every  $\zeta_1, \zeta_2, \dots, \zeta_k \in \mathcal{E}$

$$(T(\zeta_i)^* T(\zeta_j))_{k \times k} \leq (\sigma(\langle \zeta_i, \zeta_j \rangle))_{k \times k}. \quad (2.7)$$

Using the claim, the converse of part (b) can be proved in the following way:

Let  $\zeta_1, \dots, \zeta_k \in \mathcal{E}$  and  $h = (h_1, \dots, h_k), h_i \in \mathcal{H}$ . Assuming equation (2.7) holds, we observe that

$$\left\| \sum_{i=1}^k \zeta_i \otimes h_i \right\|_{\mathcal{E} \otimes \mathcal{H}}^2 = \sum_{i,j} \langle h_i, \sigma(\langle \zeta_i, \zeta_j \rangle) h_j \rangle = \langle h, (\sigma(\langle \zeta_i, \zeta_j \rangle))_{k \times k} h \rangle$$

and on the other hand

$$\begin{aligned}
\|\tilde{T}(\sum_{i=1}^k \zeta_i \otimes h_i)\|^2 &= \|\sum_{i=1}^k T(\zeta_i)h_i\|^2 \\
&= \sum_{i,j} \langle h_i, T(\zeta_i)^* T(\zeta_j)h_j \rangle \\
&= \langle h, (T(\zeta_i)^* T(\zeta_j))_{k \times k} h \rangle.
\end{aligned}$$

From the above two calculations we conclude that  $\|\tilde{T}\| \leq 1$ .

Proof of the Claim: We start with some  $\zeta_1, \dots, \zeta_k \in \mathcal{E}$ . Because we will be studying how  $T$  operates on  $\zeta_i, i = 1, \dots, d$ , we assume without loss of generality that  $\mathcal{E}$  is generated by the  $\zeta_i$ 's. A corollary of Kasparov's stabilization theorem (cf. [La95] or Appendix of chapter 6) allows us to pick vectors  $\{\eta_i\}_{i=1}^\infty$  in  $\mathcal{E}$  such that  $\{\eta_i \otimes \eta_i^*\}_{i=1}^\infty$  is a contractive approximate identity of the algebra  $K(\mathcal{E})$  of compacts. The projection  $Q := (\langle \eta_i, \eta_j \rangle)_{\infty \times \infty}$  is an element of the multiplier algebra of  $\mathcal{M} \otimes \mathfrak{K}$  (where  $\mathfrak{K}$  stands for the  $C^*$ -algebra of compact operators on any separable infinite dimensional Hilbert space). Denote  $(\sigma(\langle \eta_i, \eta_j \rangle))_{\infty \times \infty}$  (in  $B(\mathcal{H}^{(\infty)})$ ) and  $(\sigma(\langle \zeta_i, \eta_j \rangle))_{\infty \times k}$  (in  $B(\mathcal{H}^{(\infty)}, \mathcal{H}^{(n)})$ ) by  $\tilde{\sigma}(Q)$  and  $R$  respectively. We have

$$R\tilde{\sigma}(Q)R^* = (\sigma(\langle \zeta_i, \zeta_j \rangle))_{k \times k}, \quad R(T(\eta_i)^* T(\eta_j))_{\infty \times \infty} R^* = (T(\zeta_i)^* T(\zeta_j))_{k \times k}.$$

Therefore we just need to show

$$(T(\eta_i)^* T(\eta_j))_{\infty \times \infty} \leq \tilde{\sigma}(Q)$$

and this is an easy exercise on observing that

$$(T(\eta_1), T(\eta_2), \dots) \tilde{\sigma}(Q) = (T(\eta_1), T(\eta_2), \dots)$$

and employing the fact that  $T$  is completely contractive.  $\square$



## Chapter 3

# Characteristic Functions for Ergodic Tuples

**Abstract:** *Motivated by a result on weak Markov dilations, we define a notion of characteristic function for ergodic and coisometric row contractions with a one-dimensional invariant subspace for the adjoints. This extends a definition given by G. Popescu. We prove that our characteristic function is a complete unitary invariant for such tuples and show how it can be computed.*

Joint work with Rolf Gohm. Published in Integral Equations and Operator Theory 58 (2007), 43-63.

### 3.0 Introduction

If  $Z = \sum_{i=1}^d A_i \cdot A_i^*$  is a normal, unital, ergodic, completely positive map on  $B(\mathcal{H})$ , the bounded linear operators on a complex separable Hilbert space, and if there is a (necessarily unique) invariant vector state for  $Z$ , then we also say that  $\underline{A} = (A_1, \dots, A_d)$  is a coisometric, ergodic row contraction with a one-dimensional invariant subspace for the adjoints. Precise definitions are given below. This is the main setting to be investigated in this paper.

In Section 3.1 we give a concise review of a result on the dilations of  $Z$  obtained by R. Gohm in [Go04] in a chapter called ‘Cocycles and Coboundaries’. There exists a conjugacy between a homomorphic dilation of  $Z$  and a tensor shift, and we emphasize an explicit infinite product formula that can be obtained for the intertwining unitary. [Go04] may also be consulted for connections of this topic to a scattering theory for noncommutative Markov chains by B. Kümmerer and H. Maassen (cf. [KM00]) and more general for the relevance of this setting in applications.

In this work we are concerned with its relevance in operator theory and correspondingly in Section 3.2 we shift our attention to the row contraction  $\underline{A} = (A_1, \dots, A_d)$ . Our starting point has been the observation that the intertwining unitary mentioned above has many similarities with the notion of characteristic function occurring in the theory of functional models of contractions, as initiated by B. Sz.-Nagy and C. Foias (cf. [NF70, FF90]). In fact, the center of our work is the commuting diagram 3.5 in Section 3.3, which connects the results in [Go04] mentioned above with the theory of minimal isometric dilations of row contractions by G. Popescu (cf. [Po89a]) and shows that the intertwining unitary determines a multi-analytic inner function, in the sense introduced by G. Popescu in [Po89c, Po95]. We call this inner function the *extended*

*characteristic function* of the tuple  $\underline{A}$ , see Definition 3.3.3.

Section 3.4 is concerned with an explicit computation of this inner function. In Section 3.5 we show that it is an extension of the characteristic function of the  $*$ -stable part  $\overset{\circ}{\underline{A}}$  of  $\underline{A}$ , the latter in the sense of Popescu's generalization of the Sz.-Nagy-Foias theory to row contractions (cf. [Po89b]). This explains why we call our inner function an *extended* characteristic function. The row contraction  $\underline{A}$  is a one-dimensional extension of the  $*$ -stable row contraction  $\overset{\circ}{\underline{A}}$ , and in our analysis we separate the new part of the characteristic function from the part already given by Popescu.

G. Popescu has shown in [Po89b] that for completely non-coisometric tuples, in particular for  $*$ -stable ones, his characteristic function is a complete invariant for unitary equivalence. In Section 3.6 we prove that our extended characteristic function does the same for the tuples  $\underline{A}$  described above. In this sense it is *characteristic*. This is remarkable because the strength of Popescu's definition lies in the completely non-coisometric situation while we always deal with a coisometric tuple  $\underline{A}$ . The extended characteristic function also does not depend on the choice of the decomposition  $\sum_{i=1}^d A_i \cdot A_i^*$  of the completely positive map  $Z$  and hence also characterizes  $Z$  up to conjugacy. We think that together with its nice properties established earlier this clearly indicates that the extended characteristic function is a valuable tool for classifying and investigating such tuples respectively such completely positive maps.

Section 3.7 contains a worked example for the constructions in this paper.

### 3.1 Weak Markov dilations and conjugacy

In this section we give a brief and condensed review of results in [Go04], Chapter 2, which will be used in the following and which, as described in the introduction, motivated the investigations documented in this paper. We also introduce notation.

A theory of *weak Markov dilations* has been developed in [BP94]. For a (single) normal unital completely positive map  $Z : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ , where  $B(\mathcal{H})$  consists of the bounded linear operators on a (complex, separable) Hilbert space, it asks for a normal unital  $*$ -endomorphism  $\hat{J} : B(\hat{\mathcal{H}}) \rightarrow B(\hat{\mathcal{H}})$ , where  $\hat{\mathcal{H}}$  is a Hilbert space containing  $\mathcal{H}$ , such that for all  $n \in \mathbb{N}$  and all  $x \in B(\mathcal{H})$

$$Z^n(x) = p_{\mathcal{H}} \hat{J}^n(x p_{\mathcal{H}}) |_{\mathcal{H}}.$$

Here  $p_{\mathcal{H}}$  is the orthogonal projection onto  $\mathcal{H}$ . There are many ways to construct  $\hat{J}$ . In [Go04], 3.2.3, we gave a construction analogous to the idea of ‘coupling to a shift’ used in [Kü85] for describing quantum Markov processes. This gives rise to a number of interesting problems which remain hidden in other constructions.

We proceed in two steps. First note that there is a Kraus decomposition  $Z(x) = \sum_{i=1}^d a_i x a_i^*$  with  $(a_i)_{i=1}^d \subset B(\mathcal{H})$ . Here  $d = \infty$  is allowed in which case the sum should be interpreted as a limit in the strong operator topology. Let  $\mathcal{P}$  be a  $d$ -dimensional Hilbert space with orthonormal basis  $\{\epsilon_1, \dots, \epsilon_d\}$ , further  $\mathcal{K}$  another Hilbert space with a distinguished unit vector  $\Omega_{\mathcal{K}} \in \mathcal{K}$ . We identify  $\mathcal{H}$  with  $\mathcal{H} \otimes \Omega_{\mathcal{K}} \subset \mathcal{H} \otimes \mathcal{K}$  and again denote by  $p_{\mathcal{H}}$  the orthogonal projection onto  $\mathcal{H}$ . For  $\mathcal{K}$  large enough there exists an isometry

$$u : \mathcal{H} \otimes \mathcal{P} \rightarrow \mathcal{H} \otimes \mathcal{K} \quad \text{s.t.} \quad p_{\mathcal{H}} u(h \otimes \epsilon_i) = a_i(h),$$



for all  $h \in \mathcal{H}$ ,  $i = 1, \dots, d$ , or equivalently,

$$u^*(h \otimes \Omega_{\mathcal{K}}) = \sum_{i=1}^d a_i^*(h) \otimes \epsilon_i.$$

Explicitly, one may take  $\mathcal{K} = \mathbb{C}^{d+1}$  (resp. infinite-dimensional) and identify

$$\mathcal{H} \otimes \mathcal{K} \simeq (\mathcal{H} \otimes \Omega_{\mathcal{K}}) \oplus \bigoplus_1^d \mathcal{H} \simeq \mathcal{H} \oplus \bigoplus_1^d \mathcal{H}.$$

Then, using isometries  $u_1, \dots, u_d : \mathcal{H} \rightarrow \mathcal{H} \oplus \bigoplus_1^d \mathcal{H}$  with orthogonal ranges and such that  $a_i = p_{\mathcal{H}} u_i$  for all  $i$  (for example, such isometries are explicitly constructed in Popescu's formula for isometric dilations, cf. [Po89a] or equation 3.4 in Section 3), we can define

$$u(h \otimes \epsilon_i) := u_i(h)$$

for all  $h \in \mathcal{H}$ ,  $i = 1, \dots, d$  and check that  $u$  has the desired properties. Now we define a  $*$ -homomorphism

$$\begin{aligned} J : B(\mathcal{H}) &\rightarrow B(\mathcal{H} \otimes \mathcal{K}), \\ x &\mapsto u(x \otimes \mathbf{1}_{\mathcal{P}}) u^*. \end{aligned}$$

It satisfies

$$\begin{aligned} p_{\mathcal{H}} J(x)(h \otimes \Omega_{\mathcal{K}}) &= p_{\mathcal{H}} u(x \otimes \mathbf{1}) u^*(h \otimes \Omega_{\mathcal{K}}) \\ &= p_{\mathcal{H}} u(x \otimes \mathbf{1}) \left( \sum_{i=1}^d a_i^*(h) \otimes \epsilon_i \right) = \sum_{i=1}^d a_i x a_i^*(h) = Z(x)(h), \end{aligned}$$

which means that  $J$  is a kind of first order dilation for  $Z$ .

For the second step we write  $\tilde{\mathcal{K}} := \bigotimes_1^{\infty} \mathcal{K}$  for an infinite tensor product of Hilbert spaces along the sequence  $(\Omega_{\mathcal{K}})$  of unit vectors in the copies of  $\mathcal{K}$ . We have a distinguished unit vector  $\Omega_{\tilde{\mathcal{K}}}$  and a (kind of) tensor shift

$$R : B(\tilde{\mathcal{K}}) \rightarrow B(\mathcal{P} \otimes \tilde{\mathcal{K}}), \quad \tilde{y} \mapsto \mathbf{1}_{\mathcal{P}} \otimes \tilde{y}.$$

Finally  $\tilde{\mathcal{H}} := \mathcal{H} \otimes \tilde{\mathcal{K}}$  and we define a normal  $*$ -endomorphism

$$\begin{aligned} \tilde{J} : B(\tilde{\mathcal{H}}) &\rightarrow B(\tilde{\mathcal{H}}), \\ B(\mathcal{H}) \otimes B(\tilde{\mathcal{K}}) \ni x \otimes \tilde{y} &\mapsto J(x) \otimes \tilde{y} \in B(\mathcal{H} \otimes \mathcal{K}) \otimes B(\tilde{\mathcal{K}}). \end{aligned}$$

Here we used von Neumann tensor products and (on the right hand side) a shift identification  $\mathcal{K} \otimes \tilde{\mathcal{K}} \simeq \tilde{\mathcal{K}}$ . We can also write  $\tilde{J}$  in the form

$$\tilde{J}(\cdot) = u (Id_{\mathcal{H}} \otimes R)(\cdot) u^*,$$

where  $u$  is identified with  $u \otimes \mathbf{1}_{\tilde{\mathcal{K}}}$ . The natural embedding  $\mathcal{H} \simeq \mathcal{H} \otimes \Omega_{\tilde{\mathcal{K}}} \subset \tilde{\mathcal{H}}$  leads to the restriction  $\hat{J} := \tilde{J}|_{\hat{\mathcal{H}}}$  with  $\hat{\mathcal{H}} := \overline{\text{span}}_{n \geq 0} \tilde{J}^n(p_{\mathcal{H}})(\tilde{\mathcal{H}})$ , which can be checked to be a normal unital  $*$ -endomorphism satisfying all the properties of a weak Markov dilation for  $Z$  described above. See [Go04], 2.3.

A Kraus decomposition of  $\hat{J}$  can be written as

$$\hat{J}(x) = \sum_{i=1}^d t_i x t_i^*,$$

where  $t_i \in B(\hat{\mathcal{H}})$  is obtained by linear extension of  $\mathcal{H} \otimes \tilde{\mathcal{K}} \ni h \otimes \tilde{k} \mapsto u_i(h) \otimes \tilde{k} = u(h \otimes \epsilon_i) \otimes \tilde{k} \in (\mathcal{H} \otimes \mathcal{K}) \otimes \tilde{\mathcal{K}} \simeq \mathcal{H} \otimes \tilde{\mathcal{K}}$ . Because  $\hat{J}$  is a normal unital  $*$ -endomorphism the  $(t_i)_{i=1}^d$  generate a representation of the Cuntz algebra  $\mathcal{O}_d$  on  $\hat{\mathcal{H}}$  which we called a *coupling representation* in [Go04], 2.4. Note that the tuple  $(t_1, \dots, t_d)$  is an isometric dilation of the tuple  $(a_1, \dots, a_d)$ , i.e., the  $t_i$  are isometries with orthogonal ranges and  $p_{\mathcal{H}} t_i^n|_{\mathcal{H}} = a_i^n$  for all  $i = 1, \dots, d$  and  $n \in \mathbb{N}$ .

The following *multi-index notation* will be used frequently in this work. Let  $\Lambda$  denote the set  $\{1, 2, \dots, d\}$ . For operator tuples  $(a_1, \dots, a_d)$ , given  $\alpha = (\alpha_1, \dots, \alpha_m)$  in  $\Lambda^m$ ,  $a_{\alpha}$  will stand for the operator  $a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_m}$ ,  $|\alpha| := m$ . Further  $\tilde{\Lambda} := \cup_{n=0}^{\infty} \Lambda^n$ , where  $\Lambda^0 := \{0\}$  and  $a_0$  is the identity

operator. If we write  $a_\alpha^*$  this always means  $(a_\alpha)^* = a_{\alpha_m}^* \dots a_{\alpha_1}^*$ .

Back to our isometric dilation, it can be checked that

$$\overline{\text{span}}\{t_\alpha h : h \in \mathcal{H}, \alpha \in \tilde{\Lambda}\} = \hat{\mathcal{H}},$$

which means that we have a *minimal isometric dilation*, cf. [Po89a] or the beginning of Section 3.3. For more details on the construction above see [Go04], 2.3 and 2.4.

Assume now that there is an invariant vector state for  $Z : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  given by a unit vector  $\Omega_{\mathcal{H}} \in \mathcal{H}$ . Equivalent: There is a unit vector  $\Omega_{\mathcal{P}} = \sum_{i=1}^d \bar{\omega}_i \epsilon_i \in \mathcal{P}$  such that  $u(\Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{P}}) = \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}$ . Also equivalent: For  $i = 1, \dots, d$  we have  $a_i^* \Omega_{\mathcal{H}} = \bar{\omega}_i \Omega_{\mathcal{H}}$ . Here  $\omega_i \in \mathbb{C}$  with  $\sum_{i=1}^d |\omega_i|^2 = 1$  and we used complex conjugation to get nice formulas later. See [Go04], A.5.1, for a proof of the equivalences.

On  $\tilde{\mathcal{P}} := \bigotimes_1^\infty \mathcal{P}$  along the unit vectors  $(\Omega_{\mathcal{P}})$  in the copies of  $\mathcal{P}$  we have a tensor shift

$$S : B(\tilde{\mathcal{P}}) \rightarrow B(\tilde{\mathcal{P}}), \quad \tilde{y} \mapsto \mathbf{1}_{\mathcal{P}} \otimes \tilde{y}.$$

Its Kraus decomposition is  $S(\tilde{y}) = \sum_{i=1}^d s_i \tilde{y} s_i^*$  with  $s_i \in B(\tilde{\mathcal{P}})$  and  $s_i(\tilde{k}) = \epsilon_i \otimes \tilde{k}$  for  $\tilde{k} \in \tilde{\mathcal{P}}$  and  $i = 1, \dots, d$ . In [Go04], 2.5, we obtained an interesting description of the situation when the dilation  $\hat{J}$  is conjugate to the shift endomorphism  $S$ . This result will be further analyzed in this paper. We give a version suitable for our present needs but the reader should have no problems to obtain a proof of the following from [Go04], 2.5.

**Theorem 3.1.1.** *Let  $Z : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  be a normal unital completely positive map with an invariant vector state  $\langle \Omega_{\mathcal{H}}, \cdot \Omega_{\mathcal{H}} \rangle$ . Notation as introduced above,  $d \geq 2$ . The following assertions are equivalent:*

- (a)  *$Z$  is ergodic, i.e., the fixed point space of  $Z$  consists of multiples of the identity.*

(b) The vector state  $\langle \Omega_{\mathcal{H}}, \cdot \Omega_{\mathcal{H}} \rangle$  is absorbing for  $Z$ , i.e., if  $n \rightarrow \infty$  then  $\phi(Z^n(x)) \rightarrow \langle \Omega_{\mathcal{H}}, x \Omega_{\mathcal{H}} \rangle$  for all normal states  $\phi$  and all  $x \in B(\mathcal{H})$ . (In particular, the invariant vector state is unique.)

(c)  $\hat{J}$  and  $S$  are conjugate, i.e., there exists a unitary  $\mathbf{w} : \hat{\mathcal{H}} \rightarrow \tilde{\mathcal{P}}$  such that

$$\hat{J}(\hat{x}) = \mathbf{w}^* S(\mathbf{w} \hat{x} \mathbf{w}^*) \mathbf{w}.$$

(d) The  $\mathcal{O}_d$ -representations corresponding to  $\hat{J}$  and  $S$  are unitarily equivalent, i.e.,

$$\mathbf{w} t_i = s_i \mathbf{w} \quad \text{for } i = 1, \dots, d.$$

An explicit formula can be given for an intertwining unitary as occurring in (c) and (d). If any of the assertions above is valid then the following limit exists strongly,

$$\tilde{\mathbf{w}} = \lim_{n \rightarrow \infty} u_{0n}^* \dots u_{01}^* : \mathcal{H} \otimes \tilde{\mathcal{K}} \rightarrow \mathcal{H} \otimes \tilde{\mathcal{P}},$$

where we used a leg notation, i.e.,  $u_{0n} = (Id_{\mathcal{H}} \otimes R)^{n-1}(u)$ . In other words  $u_{0n}$  is  $u$  acting on  $\mathcal{H}$  and on the  $n$ -th copy of  $\mathcal{P}$ . Further  $\tilde{\mathbf{w}}$  is a partial isometry with initial space  $\hat{\mathcal{H}}$  and final space  $\tilde{\mathcal{P}} \simeq \Omega_{\mathcal{H}} \otimes \tilde{\mathcal{P}} \subset \mathcal{H} \otimes \tilde{\mathcal{P}}$  and we can define  $\mathbf{w}$  as the corresponding restriction of  $\tilde{\mathbf{w}}$ .

To illustrate the product formula for  $\mathbf{w}$ , which will be our main interest in this work, we use it to derive (d).

$$\begin{aligned} \mathbf{w} t_i(h \otimes \tilde{k}) &= \mathbf{w} [u(h \otimes \epsilon_i) \otimes \tilde{k}] = \lim_{n \rightarrow \infty} u_{0n}^* \dots u_{01}^* u_{01}(h \otimes \epsilon_i \otimes \tilde{k}) \\ &= \lim_{n \rightarrow \infty} u_{0n}^* \dots u_{02}^*(h \otimes \epsilon_i \otimes \tilde{k}) = s_i \mathbf{w}(h \otimes \tilde{k}). \end{aligned}$$

Let us finally note that Theorem 3.1.1 is related to the conjugacy results in [Pow88] and [BJP96]. Compare also Proposition 3.2.4.

### 3.2 Ergodic coisometric row contractions

In the previous section we considered a map  $Z : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  given by  $Z(x) = \sum_{i=1}^d A_i x A_i^*$ , where  $(A_i)_{i=1}^d \subset B(\mathcal{H})$ . We can think of  $(A_i)_{i=1}^d$  as a  $d$ -tuple  $\underline{A} = (A_1, \dots, A_d)$  or (with the same notation) as a linear map

$$\underline{A} = (A_1, \dots, A_d) : \bigoplus_{i=1}^d \mathcal{H} \rightarrow \mathcal{H}.$$

(Concentrating now on the tuple we have changed to capital letters  $A$ . We will sometimes return to lower case letters  $a$  when we want to emphasize that we are in the (tensor product) setting of Section 3.1.) We have the following dictionary.

$$\begin{aligned} Z(\mathbf{1}) \leq \mathbf{1} &\Leftrightarrow \sum_{i=1}^d A_i A_i^* \leq \mathbf{1} \\ &\Leftrightarrow \underline{A} \text{ is a contraction} \end{aligned}$$

$$\begin{aligned} Z(\mathbf{1}) = \mathbf{1} &\Leftrightarrow \sum_{i=1}^d A_i A_i^* = \mathbf{1} \\ (Z \text{ is called unital}) &\quad (\underline{A} \text{ is called coisometric}) \end{aligned}$$

$$\begin{aligned} \langle \Omega_{\mathcal{H}}, \cdot \Omega_{\mathcal{H}} \rangle = \langle \Omega_{\mathcal{H}}, Z(\cdot) \Omega_{\mathcal{H}} \rangle &\Leftrightarrow A_i^* \Omega_{\mathcal{H}} = \bar{\omega}_i \Omega_{\mathcal{H}}, \omega_i \in \mathbb{C}, \sum_{i=1}^d |\omega_i|^2 = 1 \\ (\text{invariant vector state}) &\quad (\text{common eigenvector for adjoints}) \end{aligned}$$

$$\begin{aligned} Z \text{ ergodic} &\Rightarrow \{A_i, A_i^*\}' = \mathbb{C} \mathbf{1} \\ (\text{trivial fixed point space}) &\quad (\text{trivial commutant}) \end{aligned}$$

The converse of the implication at the end of the dictionary is not valid. This is related to the fact that the fixed point space of a completely

positive map is not always an algebra. Compare the detailed discussion of this phenomenon in [BJKW00].

By a slight abuse of language we call the tuple (or row contraction)  $\underline{A} = (A_1, \dots, A_d)$  *ergodic* if the corresponding map  $Z$  is ergodic. With this terminology we can interpret Theorem 3.1.1 as a result about ergodic coisometric row contractions  $\underline{A}$  with a common eigenvector  $\Omega_{\mathcal{H}}$  for the adjoints  $A_i^*$ . This will be examined starting with Section 3.3. To represent these objects more explicitly let us write  $\overset{\circ}{\mathcal{H}} := \mathcal{H} \ominus \mathbb{C} \Omega_{\mathcal{H}}$ . With respect to the decomposition  $\mathcal{H} = \mathbb{C} \Omega_{\mathcal{H}} \oplus \overset{\circ}{\mathcal{H}}$  we get  $2 \times 2$ -block matrices

$$A_i = \begin{pmatrix} \omega_i & 0 \\ |\ell_i\rangle & \overset{\circ}{A}_i \end{pmatrix}, \quad A_i^* = \begin{pmatrix} \overline{\omega}_i & \langle \ell_i | \\ 0 & \overset{\circ}{A}_i^* \end{pmatrix}. \quad (3.1)$$

Here  $\overset{\circ}{A}_i \in B(\overset{\circ}{\mathcal{H}})$  and  $\ell_i \in \overset{\circ}{\mathcal{H}}$ . For the off-diagonal terms we used a Dirac notation that should be clear without further comments.

Note that the case  $d = 1$  is rather uninteresting in this setting because if  $A$  is a coisometry with block matrix  $\begin{pmatrix} \omega & 0 \\ |\ell\rangle & \overset{\circ}{A} \end{pmatrix}$  then because

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A A^* = \begin{pmatrix} |\omega|^2 & \omega \langle \ell | \\ \overline{\omega} |\ell\rangle & |\ell\rangle \langle \ell| + \overset{\circ}{A} \overset{\circ}{A}^* \end{pmatrix}$$

we always have  $\ell = 0$ . But for  $d \geq 2$  there are many interesting examples arising from unital ergodic completely positive maps with invariant vector states. See Section 3.1 and also Section 3.7 for an explicit example. We always assume  $d \geq 2$ .

**Proposition 3.2.1.** *A coisometric row contraction  $\underline{A} = (A_1, \dots, A_d)$  is ergodic with common eigenvector  $\Omega_{\mathcal{H}}$  for the adjoints  $A_1^*, \dots, A_d^*$  if and only if  $\overset{\circ}{\mathcal{H}}$  is invariant for  $A_1, \dots, A_d$  and the restricted row contraction  $(\overset{\circ}{A}_1, \dots, \overset{\circ}{A}_d)$  on  $\overset{\circ}{\mathcal{H}}$  is  $*$ -stable, i.e., for all  $h \in \overset{\circ}{\mathcal{H}}$*

$$\lim_{n \rightarrow \infty} \sum_{|\alpha|=n} \|\overset{\circ}{A}_{\alpha}^* h\|^2 = 0.$$

Here we used the multi-index notation introduced in Section 3.1. Note that  $*$ -stable tuples are also called pure, we prefer the terminology from [FF90].

*Proof.* It is clear that  $\Omega_{\mathcal{H}}$  is a common eigenvector for the adjoints if and only if  $\overset{\circ}{\mathcal{H}}$  is invariant for  $A_1, \dots, A_d$ . Let  $Z(\cdot) = \sum_{i=1}^d A_i \cdot A_i^*$  be the associated completely positive map. With  $q := \mathbf{1} - |\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{H}}|$ , the orthogonal projection onto  $\overset{\circ}{\mathcal{H}}$ , and by using  $q A_i q = A_i q \simeq \overset{\circ}{A}_i$  for all  $i$ , we get

$$Z^n(q) = \sum_{|\alpha|=n} A_{\alpha} q A_{\alpha}^* = \sum_{|\alpha|=n} \overset{\circ}{A}_{\alpha} \overset{\circ}{A}_{\alpha}^*$$

and thus for all  $h \in \overset{\circ}{\mathcal{H}}$

$$\sum_{|\alpha|=n} \|\overset{\circ}{A}_{\alpha}^* h\|^2 = \langle h, Z^n(q) h \rangle.$$

Now it is well known that ergodicity of  $Z$  is equivalent to  $Z^n(q) \rightarrow 0$  for  $n \rightarrow \infty$  in the weak operator topology. See [GKL06], Prop. 3.3.2. This completes the proof.  $\square$

**Remark 3.2.2.** Given a coisometric row contraction  $\underline{a} = (a_1, \dots, a_d)$  we also have the isometry  $u : \mathcal{H} \otimes \mathcal{P} \rightarrow \mathcal{H} \otimes \mathcal{K}$  from Section 3.1. We introduce the linear map  $a : \mathcal{P} \rightarrow B(\mathcal{H})$ ,  $k \mapsto a_k$  defined by

$$a_k^*(h) \otimes k := (\mathbf{1}_{\mathcal{H}} \otimes |k\rangle\langle k|) u^*(h \otimes \Omega_{\mathcal{K}}).$$

Compare [Go04], A.3.3. In particular  $a_i = a_{\epsilon_i}$  for  $i = 1, \dots, d$ , where  $\{\epsilon_1, \dots, \epsilon_d\}$  is the orthonormal basis of  $\mathcal{P}$  used in the definition of  $u$ . Arveson's metric operator spaces, cf. [Ar03], give a conceptual foundation for basis transformations in the operator space linearly spanned by the  $a_i$ . Similarly, in our formalism a unitary in  $B(\mathcal{P})$  transforms  $\underline{a} = (a_1, \dots, a_d)$

into another tuple  $\underline{a}' = (a'_1, \dots, a'_d)$ . If  $\Omega_{\mathcal{H}}$  is a common eigenvector for the adjoints  $a_i^*$  then  $\Omega_{\mathcal{H}}$  is also a common eigenvector for the adjoints  $(a'_i)^*$  but of course the eigenvalues are transformed to another tuple  $\underline{\omega}' = (\omega'_1, \dots, \omega'_d)$ . We should consider the tuples  $\underline{a}$  and  $\underline{a}'$  to be essentially the same. This also means that the complex numbers  $\omega_i$  are not particularly important and they should not play a role in classification. They just reflect a certain choice of orthonormal basis in the relevant metric operator space. Independent of basis transformations is the vector  $\Omega_{\mathcal{P}} = \sum_{i=1}^d \bar{\omega}_i \epsilon_i \in \mathcal{P}$  satisfying  $u(\Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{P}}) = \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}$  (see Section 3.1) and the operator  $a_{\Omega_{\mathcal{P}}} = \sum_{i=1}^d \bar{\omega}_i a_i$ .

For later use we show

**Proposition 3.2.3.** *Let  $\underline{A} = (A_1, \dots, A_d)$  be an ergodic coisometric row contraction such that  $A_i^* \Omega_{\mathcal{H}} = \bar{\omega}_i \Omega_{\mathcal{H}}$  for all  $i$ , further  $A_{\Omega_{\mathcal{P}}} := \sum_{i=1}^d \bar{\omega}_i A_i$ . Then for  $n \rightarrow \infty$  in the strong operator topology*

$$(A_{\Omega_{\mathcal{P}}}^*)^n \rightarrow |\Omega_{\mathcal{H}}\rangle \langle \Omega_{\mathcal{H}}|.$$

*Proof.* We use the setting of Section 3.1 to be able to apply Theorem 3.1.1. From  $u^*(h \otimes \Omega_{\mathcal{K}}) = \sum_{i=1}^d a_i^*(h) \otimes \epsilon_i$  we obtain

$$u^*(h \otimes \Omega_{\mathcal{K}}) = a_{\Omega_{\mathcal{P}}}^*(h) \otimes \Omega_{\mathcal{P}} \oplus h'$$

with  $h' \in \mathcal{H} \otimes \Omega_{\mathcal{P}}^\perp$ . Assume that  $h \in \overset{\circ}{\mathcal{H}}$ . Because  $u^*$  is isometric on  $\mathcal{H} \otimes \Omega_{\mathcal{K}}$  we conclude that

$$u^*(\Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}) = \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{P}} \perp u^*(h \otimes \Omega_{\mathcal{K}}) \quad (3.2)$$

and thus also  $a_{\Omega_{\mathcal{P}}}^*(h) \in \overset{\circ}{\mathcal{H}}$ . In other words,

$$a_{\Omega_{\mathcal{P}}}^*(\overset{\circ}{\mathcal{H}}) \subset \overset{\circ}{\mathcal{H}}.$$



Let  $q_n$  be the orthogonal projection from  $\mathcal{H} \otimes \bigotimes_1^n \mathcal{P}$  onto  $\Omega_{\mathcal{H}} \otimes \bigotimes_1^n \mathcal{P}$ . From Theorem 3.1.1 it follows that

$$(\mathbf{1} - q_n)u_{0n}^* \dots u_{01}^*(h \otimes \bigotimes_1^n \Omega_{\mathcal{K}}) \rightarrow 0 \quad (n \rightarrow \infty).$$

On the other hand, by iterating the formula from the beginning,

$$u_{0n}^* \dots u_{01}^*(h \otimes \bigotimes_1^n \Omega_{\mathcal{K}}) = ((a_{\Omega_{\mathcal{P}}}^*)^n(h) \otimes \bigotimes_1^n \Omega_{\mathcal{P}}) \oplus h'$$

with  $h' \in \mathcal{H} \otimes (\bigotimes_1^n \Omega_{\mathcal{P}})^\perp$ . It follows that also

$$(\mathbf{1} - q_n)((a_{\Omega_{\mathcal{P}}}^*)^n(h) \otimes \bigotimes_1^n \Omega_{\mathcal{P}}) \rightarrow 0.$$

But from  $a_{\Omega_{\mathcal{P}}}^*(\overset{\circ}{\mathcal{H}}) \subset \overset{\circ}{\mathcal{H}}$  we have  $q_n((a_{\Omega_{\mathcal{P}}}^*)^n(h) \otimes \bigotimes_1^n \Omega_{\mathcal{P}}) = 0$  for all  $n$ . We conclude that  $(a_{\Omega_{\mathcal{P}}}^*)^n(h) \rightarrow 0$  for  $n \rightarrow \infty$ . Further

$$a_{\Omega_{\mathcal{P}}}^* \Omega_{\mathcal{H}} = \sum_{i=1}^d \omega_i a_i^* \Omega_{\mathcal{H}} = \sum_{i=1}^d \omega_i \overline{\omega}_i \Omega_{\mathcal{H}} = \Omega_{\mathcal{H}},$$

and the proposition is proved.  $\square$

The following proposition summarizes some well known properties of minimal isometric dilations and associated Cuntz algebra representations.

**Proposition 3.2.4.** *Suppose  $\underline{A}$  is a coisometric tuple on  $\mathcal{H}$  and  $\underline{V}$  is its minimal isometric dilation. Assume  $\Omega_{\mathcal{H}}$  is a distinguished unit vector in  $\mathcal{H}$  and  $\underline{\omega} = (\omega_1, \dots, \omega_d) \in \mathbb{C}^d$ ,  $\sum_i |\omega_i|^2 = 1$ . Then the following are equivalent.*

1.  $\underline{A}$  is ergodic and  $A_i^* \Omega_{\mathcal{H}} = \overline{\omega}_i \Omega_{\mathcal{H}}$  for all  $i$ .
2.  $\underline{V}$  is ergodic and  $V_i^* \Omega_{\mathcal{H}} = \overline{\omega}_i \Omega_{\mathcal{H}}$  for all  $i$ .

3.  $V_i^* \Omega_{\mathcal{H}} = \overline{\omega}_i \Omega_{\mathcal{H}}$  and  $\underline{V}$  generates the GNS-representation of the Cuntz algebra  $\mathcal{O}_d = C^*\{g_1, \dots, g_d\}$  ( $g_i$  its abstract generators) with respect to the Cuntz state which maps

$$g_\alpha g_\beta^* \mapsto \omega_\alpha \overline{\omega}_\beta, \quad \forall \alpha, \beta \in \tilde{\Lambda}.$$

*Cuntz states are pure and the corresponding GNS-representations are irreducible.*

This Proposition clearly follows from Theorem 5.1 of [BJKW00], Theorem 3.3 and Theorem 4.1 of [BJP96]. Note that in Theorem 3.1.1(d) we already saw a concrete version of the corresponding Cuntz algebra representation.

### 3.3 A new characteristic function

First we recall some more details of the theory of minimal isometric dilations for row contractions (cf. [Po89a]) and introduce further notation.

The full Fock space over  $\mathbb{C}^d$  ( $d \geq 2$ ) denoted by  $\Gamma(\mathbb{C}^d)$  is

$$\Gamma(\mathbb{C}^d) := \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes 2} \oplus \dots \oplus (\mathbb{C}^d)^{\otimes m} \oplus \dots.$$

$1 \oplus 0 \oplus \dots$  is called the vacuum vector. Let  $\{e_1, \dots, e_d\}$  be the standard orthonormal basis of  $\mathbb{C}^d$ . Recall that we include  $d = \infty$  in which case  $\mathbb{C}^d$  stands for a complex separable Hilbert space of infinite dimension. For  $\alpha \in \tilde{\Lambda}$ ,  $e_\alpha$  will denote the vector  $e_{\alpha_1} \otimes e_{\alpha_2} \otimes \dots \otimes e_{\alpha_m}$  in the full Fock space  $\Gamma(\mathbb{C}^d)$  and  $e_0$  will denote the vacuum vector. Then the (left) creation operators  $L_i$  on  $\Gamma(\mathbb{C}^d)$  are defined by

$$L_i x = e_i \otimes x$$

for  $1 \leq i \leq d$  and  $x \in \Gamma(\mathbb{C}^d)$ . The row contraction  $\underline{L} = (L_1, \dots, L_d)$  consists of isometries with orthogonal ranges.

Let  $\underline{T} = (T_1, \dots, T_d)$  be a row contraction on a Hilbert space  $\mathcal{H}$ . Treating  $\underline{T}$  as a row operator from  $\bigoplus_{i=1}^d \mathcal{H}$  to  $\mathcal{H}$ , define  $D_* := (\mathbf{1} - \underline{T}\underline{T}^*)^{\frac{1}{2}} : \mathcal{H} \rightarrow \mathcal{H}$  and  $D := (\mathbf{1} - \underline{T}^*\underline{T})^{\frac{1}{2}} : \bigoplus_{i=1}^d \mathcal{H} \rightarrow \bigoplus_{i=1}^d \mathcal{H}$ . This implies that

$$D_* = (\mathbf{1} - \sum_{i=1}^d T_i T_i^*)^{\frac{1}{2}}, \quad D = (\delta_{ij} \mathbf{1} - T_i^* T_j)^{\frac{1}{2}}_{d \times d}. \quad (3.3)$$

Observe that  $\underline{T}D^2 = D_*^2 \underline{T}$  and hence  $\underline{T}D = D_* \underline{T}$ . Let  $\mathcal{D} := \text{Range } D$  and  $\mathcal{D}_* := \text{Range } D_*$ . Popescu in [Po89a] gave the following explicit presentation of the minimal isometric dilation of  $\underline{T}$  by  $\underline{V}$  on  $\mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D})$ ,

$$V_i(h \oplus \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha) = T_i h \oplus [e_0 \otimes D_i h + e_i \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha] \quad (3.4)$$

for  $h \in \mathcal{H}$  and  $d_\alpha \in \mathcal{D}$ . Here  $D_i h := D(0, \dots, 0, h, 0, \dots, 0)$  and  $h$  is embedded at the  $i^{\text{th}}$  component.

In other words, the  $V_i$  are isometries with orthogonal ranges such that  $T_i^* = V_i^*|_{\mathcal{H}}$  for  $i = 1, \dots, d$  and the spaces  $V_\alpha \mathcal{H}$  with  $\alpha \in \tilde{\Lambda}$  together span the Hilbert space on which the  $V_i$  are defined. It is an important fact, which we shall use repeatedly, that such minimal isometric dilations are unique up to unitary equivalence (cf. [Po89a]).

Now, as in Section 3.2, let  $\underline{A} = (A_1, \dots, A_d)$ ,  $A_i \in B(\mathcal{H})$ , be an ergodic coisometric tuple with  $A_i^* \Omega_{\mathcal{H}} = \bar{\omega}_i \Omega_{\mathcal{H}}$  for some unit vector  $\Omega_{\mathcal{H}} \in \mathcal{H}$  and some  $\underline{\omega} \in \mathbb{C}^d$ ,  $\sum_i |\omega_i|^2 = 1$ . Let  $\underline{V} = (V_1, \dots, V_d)$  be the minimal isometric dilation of  $\underline{A}$  given by Popescu's construction (see equation 3.4) on  $\mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_A)$ . Because  $A_i^* = V_i^*|_{\mathcal{H}}$  we also have  $V_i^* \Omega_{\mathcal{H}} = \bar{\omega}_i \Omega_{\mathcal{H}}$  and because  $\underline{V}$  generates an irreducible  $\mathcal{O}_d$ -representation (Proposition 3.2.4), we see that  $\underline{V}$  is also a minimal isometric dilation of  $\underline{\omega} : \mathbb{C}^d \rightarrow \mathbb{C}$ .

In fact, we can think of  $\underline{\omega}$  as the most elementary example of a tuple with all the properties stated for  $\underline{A}$ . Let  $\tilde{V} = (\tilde{V}_1, \dots, \tilde{V}_d)$  be the minimal isometric dilation of  $\underline{\omega}$  given by Popescu's construction on  $\mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega)$ .

Because  $\underline{A}$  is coisometric it follows from equation 3.3 that  $D$  is in fact a projection and hence  $D = (\delta_{ij} \mathbf{1} - A_i^* A_j)_{d \times d}$ . We infer that  $D(A_1^*, \dots, A_d^*)^T = 0$ , where  $T$  stands for transpose. Applied to  $\underline{\omega}$  instead of  $\underline{A}$  this shows that  $D_\omega = (\mathbf{1} - |\underline{\omega}\rangle\langle\underline{\omega}|)$  and

$$\mathcal{D}_\omega \oplus \mathbb{C}(\overline{\omega}_1, \dots, \overline{\omega}_d)^T = \mathbb{C}^d,$$

where  $\underline{\omega} = (\overline{\omega}_1, \dots, \overline{\omega}_d)$ .

**Remark 3.3.1.** *Because  $\Omega_{\mathcal{H}}$  is cyclic for  $\{V_\alpha, \alpha \in \tilde{\Lambda}\}$  we have*

$$\overline{\text{span}}\{A_\alpha \Omega_{\mathcal{H}} : \alpha \in \tilde{\Lambda}\} = \overline{\text{span}}\{p_{\mathcal{H}} V_\alpha \Omega_{\mathcal{H}} : \alpha \in \tilde{\Lambda}\} = \mathcal{H}.$$

*Using the notation from equation 3.1 this further implies that*

$$\overline{\text{span}}\{\dot{A}_\alpha l_i : \alpha \in \tilde{\Lambda}, 1 \leq i \leq d\} = \dot{\mathcal{H}}.$$

As minimal isometric dilations of the tuple  $\underline{\omega}$  are unique up to unitary equivalence, there exists a unitary

$$W : \mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_A) \rightarrow \mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega),$$

such that  $WV_i = \tilde{V}_i W$  for all  $i$ .

After showing the existence of  $W$  we now proceed to compute  $W$  explicitly. For  $\underline{A}$ , by using Popescu's construction, we have its minimal isometric dilation  $\underline{V}$  on  $\mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_A)$ . Another way of constructing a minimal isometric dilation  $\underline{t}$  of  $\underline{a}$  was demonstrated in Section 3.1 on

the space  $\hat{\mathcal{H}}$  (obtained by restricting to the minimal subspace of  $\mathcal{H} \otimes \tilde{\mathcal{K}}$  with respect to  $\underline{t}$ ). Identifying  $\underline{A}$  and  $\underline{a}$  on the Hilbert space  $\mathcal{H}$  there is a unitary  $\Gamma_A : \hat{\mathcal{H}} \rightarrow \mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_A)$  which is the identity on  $\mathcal{H}$  and satisfies  $V_i \Gamma_A = \Gamma_A t_i$ .

By Theorem 3.1.1(d) the tuple  $\underline{s}$  on  $\tilde{\mathcal{P}}$  arising from the tensor shift is unitarily equivalent to  $\underline{t}$  (resp.  $\underline{V}$ ), explicitly  $\mathbf{w} t_i = s_i \mathbf{w}$  for all  $i$ . An alternative viewpoint on the existence of  $\mathbf{w}$  is to note that  $\underline{s}$  is a minimal isometric dilation of  $\underline{\omega}$ . In fact,  $s_i^* \Omega_{\tilde{\mathcal{P}}} = \langle \epsilon_i, \Omega_{\mathcal{P}} \rangle \Omega_{\tilde{\mathcal{P}}} = \bar{\omega}_i \Omega_{\tilde{\mathcal{P}}}$  for all  $i$ . Hence there is also a unitary  $\Gamma_{\omega} : \tilde{\mathcal{P}} \rightarrow \mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_{\omega})$  with  $\Gamma_{\omega} \Omega_{\tilde{\mathcal{P}}} = 1 \in \mathbb{C}$  which satisfies  $\tilde{V}_i \Gamma_{\omega} = \Gamma_{\omega} s_i$ .

**Remark 3.3.2.** *It is possible to describe  $\Gamma_{\omega}$  in an explicit way and in doing so to construct an interesting and natural (unitary) identification of  $\bigotimes_1^{\infty} \mathbb{C}^d$  and  $\mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathbb{C}^{d-1})$ . In fact, recall (from Section 3.1) that  $\tilde{\mathcal{P}} = \bigotimes_1^{\infty} \mathcal{P}$  and the space  $\mathcal{P}$  is nothing but a  $d$ -dimensional Hilbert space. Hence we can identify*

$$\mathbb{C}^d \simeq \mathcal{P} = \overset{\circ}{\tilde{\mathcal{P}}} \oplus \mathbb{C} \Omega_{\mathcal{P}} \simeq \mathcal{D}_{\omega} \oplus \mathbb{C} \underline{\omega}^T \simeq \mathbb{C}^{d-1} \oplus \mathbb{C}$$

*In this identification the orthonormal basis  $(\epsilon_i)_{i=1}^d$  of  $\mathcal{P}$  goes to the canonical basis  $(e_i)_{i=1}^d$  of  $\mathbb{C}^d$ , in particular the vector  $\Omega_{\mathcal{P}} = \sum_i \bar{\omega}_i \epsilon_i$  goes to  $\underline{\omega}^T = (\bar{\omega}_1, \dots, \bar{\omega}_d)^T$  and we have  $\overset{\circ}{\tilde{\mathcal{P}}} \simeq \mathcal{D}_{\omega}$ . Then we can write*

$$\begin{aligned} \Gamma_{\omega} : \quad \Omega_{\tilde{\mathcal{P}}} &\mapsto 1 \in \mathbb{C}, \\ k \otimes \Omega_{\tilde{\mathcal{P}}} &\mapsto e_0 \otimes k \\ \epsilon_{\alpha} \otimes k \otimes \Omega_{\tilde{\mathcal{P}}} &\mapsto e_{\alpha} \otimes k, \end{aligned}$$

*where  $k \in \overset{\circ}{\tilde{\mathcal{P}}}$ ,  $\alpha \in \tilde{\Lambda}$ ,  $\epsilon_{\alpha} = \epsilon_{\alpha_1} \otimes \dots \otimes \epsilon_{\alpha_n} \in \bigotimes_1^n \mathcal{P}$  (the first  $n$  copies of  $\mathcal{P}$  in the infinite tensor product  $\tilde{\mathcal{P}}$ ),  $e_{\alpha} = e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n} \in \Gamma(\mathbb{C}^d)$  as usual.*

It is easily checked that  $\Gamma_\omega$  given in this way indeed satisfies the equation  $\tilde{V}_i \Gamma_\omega = \Gamma_\omega s_i$  (for all  $i$ ), which may thus be seen as the abstract characterization of this unitary map (together with  $\Gamma_\omega \Omega_{\tilde{\mathcal{P}}} = 1$ ).

Summarizing, for  $i = 1, \dots, d$

$$V_i \Gamma_A = \Gamma_A t_i, \quad \mathbf{w} t_i = s_i \mathbf{w}, \quad \tilde{V}_i \Gamma_\omega = \Gamma_\omega s_i$$

and we have the commuting diagram

$$\begin{array}{ccc} \hat{\mathcal{H}} & \xrightarrow{\mathbf{w}} & \tilde{\mathcal{P}} \\ \Gamma_A \downarrow & & \downarrow \Gamma_\omega \\ \mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_A) & \xrightarrow{W} & \mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega). \end{array} \quad (3.5)$$

From the diagram we get

$$W = \Gamma_\omega \mathbf{w} \Gamma_A^{-1}.$$

Combined with the equations above this yields  $W V_i = \tilde{V}_i W$  and we see that  $W$  is nothing but the dilations-intertwining map which we have already introduced earlier. Hence  $\mathbf{w}$  and  $W$  are essentially the same thing and for the study of certain problems it may be helpful to switch from one picture to the other.

In the following we analyze  $W$  to arrive at an interpretation as a new kind of characteristic function. First we have an isometric embedding

$$\hat{C} := W|_{\mathcal{H}} : \mathcal{H} \rightarrow \mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega). \quad (3.6)$$

Note that  $\hat{C} \Omega_{\mathcal{H}} = W \Omega_{\mathcal{H}} = 1 \in \mathbb{C}$ . The remaining part is an isometry

$$M_{\hat{\Theta}} := W|_{\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_A} : \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_A \rightarrow \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega. \quad (3.7)$$

From equation 3.4 we get for all  $i$

$$V_i|_{\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_A} = (L_i \otimes \mathbf{1}_{\mathcal{D}_A}),$$

$$\tilde{V}_i|_{\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega} = (L_i \otimes \mathbf{1}_{\mathcal{D}_\omega}),$$

and we conclude that

$$M_{\hat{\Theta}}(L_i \otimes \mathbf{1}_{\mathcal{D}_A}) = (L_i \otimes \mathbf{1}_{\mathcal{D}_\omega})M_{\hat{\Theta}}, \quad \forall 1 \leq i \leq d. \quad (3.8)$$

In other words,  $M_{\hat{\Theta}}$  is a multi-analytic inner function in the sense of [Po89c, Po95]. It is determined by its symbol

$$\hat{\theta} := W|_{e_0 \otimes \mathcal{D}_A} : \mathcal{D}_A \rightarrow \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega, \quad (3.9)$$

where we have identified  $e_0 \otimes \mathcal{D}_A$  and  $\mathcal{D}_A$ . In other words, we think of the symbol  $\hat{\theta}$  as an isometric embedding of  $\mathcal{D}_A$  into  $\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega$ .

**Definition 3.3.3.** *We call  $M_{\hat{\Theta}}$  (or  $\hat{\theta}$ ) the extended characteristic function of the row contraction  $\underline{A}$ ,*

See Sections 3.5 and 3.6 for more explanation and justification of this terminology.

### 3.4 Explicit computation of the extended characteristic function

To express the extended characteristic function more explicitly in terms of the tuple  $\underline{A}$  we start by defining

$$\hat{D}_* : \overset{\circ}{\mathcal{H}} = \mathcal{H} \ominus \mathbb{C}\Omega_{\mathcal{H}} \rightarrow \overset{\circ}{\mathcal{P}} = \mathcal{P} \ominus \mathbb{C}\Omega_{\mathcal{P}} \simeq \mathcal{D}_\omega, \quad (3.10)$$

$$h \mapsto (\langle \Omega_{\mathcal{H}} | \otimes \mathbf{1}_{\mathcal{P}}) u^*(h \otimes \Omega_{\mathcal{K}}),$$

where  $u : \mathcal{H} \otimes \mathcal{P} \rightarrow \mathcal{H} \otimes \mathcal{K}$  is the isometry introduced in Section 3.1. That indeed the range of  $\hat{D}_*$  is contained in  $\overset{\circ}{\mathcal{P}}$  follows from equation 3.2, i.e.,  $u^*(h \otimes \Omega_{\mathcal{K}}) \perp \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{P}}$  for  $h \in \overset{\circ}{\mathcal{H}}$ . With notations from equation 3.1 we can get a more concrete formula.

**Lemma 3.4.1.** *For all  $h \in \mathring{\mathcal{H}}$  we have  $\hat{D}_*(h) = \sum_{i=1}^d \langle \ell_i, h \rangle \epsilon_i$ .*

*Proof.*  $(\langle \Omega_{\mathcal{H}} | \otimes \mathbf{1}_{\mathcal{P}}) u^*(h \otimes \Omega_{\mathcal{K}}) = \sum_{i=1}^d \langle \Omega_{\mathcal{H}}, a_i^* h \rangle \otimes \epsilon_i = \sum_{i=1}^d \langle \ell_i, h \rangle \epsilon_i$ .  $\square$

**Proposition 3.4.2.** *The map  $\hat{C} : \mathcal{H} \rightarrow \mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_{\omega})$  from equation 3.7 is given explicitly by  $\hat{C}\Omega_{\mathcal{H}} = 1$  and for  $h \in \mathring{\mathcal{H}}$  by*

$$\hat{C}h = \sum_{\alpha \in \tilde{\Lambda}} e_{\alpha} \otimes \hat{D}_* \hat{A}_{\alpha}^* h.$$

*Proof.* As  $W\Omega_{\mathcal{H}} = 1$  also  $\hat{C}\Omega_{\mathcal{H}} = 1$ . Assume  $h \in \mathring{\mathcal{H}}$ . Then

$$\begin{aligned} u_{01}(h \otimes \Omega_{\tilde{\mathcal{K}}}) &= \sum_i a_i^* h \otimes \epsilon_i \otimes \Omega_{\tilde{\mathcal{K}}} \\ &= \sum_i \langle \ell_i, h \rangle \Omega_{\mathcal{H}} \otimes \epsilon_i \otimes \Omega_{\tilde{\mathcal{K}}} + \sum_i \hat{a}_i^* h \otimes \epsilon_i \otimes \Omega_{\tilde{\mathcal{K}}}. \end{aligned}$$

Because  $u^*(\Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}) = \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{P}}$  we obtain (with Lemma 3.4.1) for the first part

$$\begin{aligned} &\lim_{n \rightarrow \infty} u_{0n}^* \cdots u_{02}^* \left( \sum_i \langle \ell_i, h \rangle \Omega_{\mathcal{H}} \otimes \epsilon_i \otimes \Omega_{\tilde{\mathcal{K}}} \right) \\ &= \sum_i \langle \ell_i, h \rangle \Omega_{\mathcal{H}} \otimes \epsilon_i \otimes \Omega_{\tilde{\mathcal{P}}} = \Omega_{\mathcal{H}} \otimes \hat{D}_* h \otimes \Omega_{\tilde{\mathcal{P}}} \simeq \hat{D}_* h \otimes \Omega_{\tilde{\mathcal{P}}} \in \tilde{\mathcal{P}}. \end{aligned}$$

Using the product formula from Theorem 3.1.1 and iterating the argument above we get

$$\begin{aligned} \hat{C}(h) &= Wh = \Gamma_{\omega} \mathbf{w} \Gamma_A^{-1}(h) \\ &= \Gamma_{\omega}(\hat{D}_* h \otimes \Omega_{\tilde{\mathcal{P}}}) + \Gamma_{\omega} \lim_{n \rightarrow \infty} u_{0n}^* \cdots u_{02}^* \sum_i \hat{a}_i^* h \otimes \epsilon_i \otimes \Omega_{\tilde{\mathcal{K}}} \\ &= e_0 \otimes \hat{D}_* h + \Gamma_{\omega} \lim_{n \rightarrow \infty} u_{0n}^* \cdots u_{03}^* \sum_{j,i} (\langle \ell_j, \hat{a}_i^* h \rangle \Omega_{\mathcal{H}} + \hat{a}_j^* \hat{a}_i^* h) \otimes \epsilon_i \otimes \epsilon_j \otimes \Omega_{\tilde{\mathcal{K}}} \\ &= e_0 \otimes \hat{D}_* h + \sum_{i=1}^d e_i \otimes \hat{D}_* \hat{a}_i^* h + \Gamma_{\omega} \lim_{n \rightarrow \infty} u_{0n}^* \cdots u_{03}^* \sum_{j,i} \hat{a}_j^* \hat{a}_i^* h \otimes \epsilon_i \otimes \epsilon_j \otimes \Omega_{\tilde{\mathcal{K}}} \end{aligned}$$



$$\begin{aligned}
&= \dots \\
&= \sum_{|\alpha| < m} e_\alpha \otimes \hat{D}_* \mathring{a}_\alpha^* h + \Gamma_\omega \lim_{n \rightarrow \infty} u_{0n}^* \cdots u_{0,m+1}^* \sum_{|\alpha|=m} \mathring{a}_\alpha^* h \otimes \epsilon_\alpha \otimes \Omega_{\tilde{\mathcal{K}}}.
\end{aligned}$$

From Proposition 3.2.1 we have  $\sum_{|\alpha|=m} \|\mathring{a}_\alpha^* h\|^2 \rightarrow 0$  for  $m \rightarrow \infty$  and we conclude that the last term converges to 0. It follows that the series converges and this proves Proposition 3.4.2.  $\square$

**Remark 3.4.3.** *Another way to prove Proposition 3.4.2 for  $h \in \mathring{\mathcal{H}}$  consists in repeatedly applying the formula*

$$u^*(h \otimes \Omega_{\mathcal{K}}) = a_{\Omega_{\mathcal{P}}}^* h \otimes \Omega_{\mathcal{P}} + h', \quad h' \in \mathcal{H} \otimes \mathring{\mathcal{P}}$$

to the  $u_{0n}^*(h \otimes \Omega_{\mathcal{K}})$  and then using  $(a_{\Omega_{\mathcal{P}}}^*)^n h \rightarrow 0$ , see Proposition 3.2.3. This gives some insight how the infinite product in Theorem 3.1.1 transforms into the infinite sum in Proposition 3.4.2.

Now we present an explicit computation of the extended characteristic function. One way of writing  $\mathcal{D}_A$  is

$$\mathcal{D}_A = \overline{\text{span}}\{(V_i - A_i)h : i \in \Lambda, h \in \mathcal{H}\}.$$

Let  $d_h^i := (V_i - A_i)h$ . Then

$$\hat{\theta} d_h^i = W(V_i - A_i)h = \tilde{V}_i \hat{C}h - \hat{C}A_i h.$$

**Case I:** Take  $h = \Omega_{\mathcal{H}}$ .

$$\tilde{V}_i \hat{C} \Omega_{\mathcal{H}} = \tilde{V}_i 1 = \omega_i \oplus [e_0 \otimes (\mathbf{1} - |\underline{\omega}\rangle \langle \underline{\omega}|) \epsilon_i],$$

$$\hat{C}A_i \Omega_{\mathcal{H}} = \omega_i \oplus \sum_{\alpha} e_\alpha \otimes \hat{D}_* \mathring{A}_\alpha^* l_i$$

and thus

$$\begin{aligned}
\hat{\theta} d_{\Omega_{\mathcal{H}}}^i &= e_0 \otimes [(\mathbf{1} - |\underline{\omega}\rangle\langle\underline{\omega}|)\epsilon_i - \hat{D}_* l_i] - \sum_{|\alpha| \geq 1} e_\alpha \otimes \hat{D}_* \mathring{A}_\alpha^* l_i \\
&= e_0 \otimes [\epsilon_i - \sum_j \bar{\omega}_j \omega_i \epsilon_j - \sum_j \langle l_j, l_i \rangle \epsilon_j] - \sum_{|\alpha| \geq 1} e_\alpha \otimes \sum_j \langle l_j, \mathring{A}_\alpha^* l_i \rangle \epsilon_j \\
&= e_0 \otimes [\epsilon_i - \sum_j (\bar{\omega}_j \omega_i + \langle l_j, l_i \rangle) \epsilon_j] - \sum_{|\alpha| \geq 1} e_\alpha \otimes \sum_j \langle \mathring{A}_\alpha l_j, l_i \rangle \epsilon_j \\
&= e_0 \otimes [\epsilon_i - \sum_j \langle A_j \Omega_{\mathcal{H}}, A_i \Omega_{\mathcal{H}} \rangle \epsilon_j] - \sum_{|\alpha| \geq 1} e_\alpha \otimes \sum_j \langle \mathring{A}_\alpha l_j, l_i \rangle \epsilon_j. \quad (3.11)
\end{aligned}$$

**Case II:** Now let  $h \in \mathring{\mathcal{H}}$ . With  $i \in \Lambda$

$$\tilde{V}_i \hat{C} h = (L_i \otimes \mathbf{1}) \hat{C} h = \sum_{\alpha} e_i \otimes e_\alpha \otimes \hat{D}_* \mathring{A}_\alpha^* h,$$

$$\hat{C} A_i h = \sum_{\beta} e_\beta \otimes \hat{D}_* \mathring{A}_\beta^* \mathring{A}_i h.$$

Finally

$$\begin{aligned}
\hat{\theta} d_h^i &= \sum_{\alpha} e_i \otimes e_\alpha \otimes \hat{D}_* \mathring{A}_\alpha^* h - \sum_{\beta} e_\beta \otimes \hat{D}_* \mathring{A}_\beta^* \mathring{A}_i h \\
&= -e_0 \otimes \hat{D}_* \mathring{A}_i h + e_i \otimes \sum_{\alpha} e_\alpha \otimes \hat{D}_* \mathring{A}_\alpha^* (\mathbf{1} - \mathring{A}_i^* \mathring{A}_i) h + \sum_{j \neq i} e_j \otimes \sum_{\alpha} e_\alpha \otimes \hat{D}_* \mathring{A}_\alpha^* (-\mathring{A}_j^* \mathring{A}_i) h \\
&= -e_0 \otimes \hat{D}_* \mathring{A}_i h + \sum_{j=1}^d e_j \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes \hat{D}_* \mathring{A}_\alpha^* (\delta_{ji} \mathbf{1} - \mathring{A}_j^* \mathring{A}_i) h. \quad (3.12)
\end{aligned}$$

### 3.5 Case II is Popescu's characteristic function

In this section we show that case II in the previous section can be identified with the characteristic function of the  $*$ -stable tuple  $\underline{\mathring{A}}$ , in the sense

introduced by Popescu in [Po89b]. This is the reason why we have called  $\hat{\theta}$  an *extended* characteristic function. All information about  $\underline{A}$  beyond  $\underline{\hat{A}}$  must be contained in case I.

First recall the theory of characteristic functions for row contractions, as developed by G. Popescu in [Po89b], generalizing the theory of B. Sz.-Nagy and C. Foias (cf. [NF70]) for single contractions. We only need the results about a  $*$ -stable tuple  $\underline{\hat{A}} = (\hat{A}_1, \dots, \hat{A}_d)$  on  $\hat{\mathcal{H}}$ . In this case, with  $\hat{D}_* = (1 - \underline{\hat{A}}\hat{A}^*)^{\frac{1}{2}} : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}$  and  $\hat{\mathcal{D}}_*$  its range, the map

$$\hat{C} : \hat{\mathcal{H}} \rightarrow \Gamma(\mathbb{C}^d) \otimes \hat{\mathcal{D}}_* \quad (3.13)$$

$$h \mapsto \sum_{\alpha \in \tilde{\Lambda}} e_{\alpha} \otimes \hat{D}_* \hat{A}_{\alpha}^* h$$

is an isometry (Popescu's Poisson kernel). If, as usual,  $\hat{D} = (1 - \underline{\hat{A}}^* \hat{A})^{\frac{1}{2}} : \bigoplus_1^d \hat{\mathcal{H}} \rightarrow \bigoplus_1^d \hat{\mathcal{H}}$ , with  $\hat{\mathcal{D}}$  its range, and if  $P_j$  is the projection onto the  $j$ -th component, then the characteristic function  $\theta_{\hat{A}}$  of  $\underline{\hat{A}}$  can be defined as

$$\theta_{\hat{A}} : \hat{\mathcal{D}} \rightarrow \Gamma(\mathbb{C}^d) \otimes \hat{\mathcal{D}}_* \quad (3.14)$$

$$f \mapsto -e_0 \otimes \sum_{j=1}^d \hat{A}_j P_j f + \sum_{j=1}^d e_j \otimes \sum_{\alpha \in \tilde{\Lambda}} e_{\alpha} \otimes \hat{D}_* \hat{A}_{\alpha}^* P_j \hat{D} f.$$

See [Po89b] for details, in particular for the important result that  $\theta_{\hat{A}}$  characterizes the  $*$ -stable tuple  $\underline{\hat{A}}$  up to unitary equivalence.

Now consider again the tuple  $\underline{A}$  of the previous section, with extended characteristic function  $\hat{\theta}$ . From equation 3.1

$$A_i = \begin{pmatrix} \omega_i & 0 \\ |\ell_i\rangle & \hat{A}_i \end{pmatrix}, \quad A_i^* = \begin{pmatrix} \overline{\omega_i} & \langle \ell_i| \\ 0 & \hat{A}_i^* \end{pmatrix}$$

and hence

$$A_i A_i^* = \begin{pmatrix} |\bar{\omega}_i|^2 & \langle \bar{\omega}_i l_i | \\ |\bar{\omega}_i l_i \rangle & |l_i\rangle \langle l_i| + \mathring{A}_i \mathring{A}_i^* \end{pmatrix}.$$

Recall that  $D_*^2 = \mathbf{1} - \sum_i A_i A_i^*$  which is 0 as  $\underline{A}$  is coisometric. Thus  $\sum_i \bar{\omega}_i l_i = 0$  and  $\mathbf{1} - \sum_i \mathring{A}_i \mathring{A}_i^* = \sum_i |l_i\rangle \langle l_i|$ . The first equation means that  $A_{\Omega_{\mathcal{P}}}^*(\mathring{\mathcal{H}}) \subset \mathring{\mathcal{H}}$  and that

$$\langle \hat{\mathcal{D}}_* h, \Omega_{\mathcal{P}} \rangle = \langle \sum_i \langle l_i, h \rangle \epsilon_i, \sum_j \bar{\omega}_j \epsilon_j \rangle = \langle \sum_i \bar{\omega}_i l_i, h \rangle = 0,$$

which we already know (see 3.10).

The second equation yields

$$\mathring{D}_*^2 = \mathbf{1} - \sum_i \mathring{A}_i \mathring{A}_i^* = \sum_i |l_i\rangle \langle l_i|.$$

**Lemma 3.5.1.** *There exists an isometry  $\gamma : \mathring{\mathcal{D}}_* \rightarrow \mathring{\mathcal{P}} \simeq \mathcal{D}_{\omega}$  defined for  $h \in \mathring{\mathcal{H}}$  as*

$$\mathring{D}_* h \mapsto \sum_i \langle l_i, h \rangle \epsilon_i = \hat{\mathcal{D}}_* h.$$

*Proof.* Take  $h \in \mathring{\mathcal{H}}$ . By Lemma 3.4.1 we have  $\hat{\mathcal{D}}_*(h) = \sum_{i=1}^d \langle l_i, h \rangle \epsilon_i$ . Now we can compute

$$\|\hat{\mathcal{D}}_* h\|^2 = \langle \sum_i \langle l_i, h \rangle \epsilon_i, \sum_j \langle l_j, h \rangle \epsilon_j \rangle = \sum_i \langle h, l_i \rangle \langle l_i, h \rangle = \langle h, \mathring{D}_*^2 h \rangle = \|\mathring{D}_* h\|^2.$$

Hence  $\gamma : \mathring{D}_* h \mapsto \hat{\mathcal{D}}_* h$  is isometric.  $\square$

**Theorem 3.5.2.** *Let  $\underline{A} = (A_1, \dots, A_d)$ ,  $A_i \in B(\mathcal{H})$ , be an ergodic coisometric tuple with  $A_i^* \Omega_{\mathcal{H}} = \bar{\omega}_i \Omega_{\mathcal{H}}$  for some unit vector  $\Omega_{\mathcal{H}} \in \mathcal{H}$  and some  $\omega \in \mathbb{C}^d$ ,  $\sum_i |\omega_i|^2 = 1$ . Let  $\hat{\theta}$  be the extended characteristic function of  $\underline{A}$  and let  $\theta_A^{\circ}$  be the characteristic function of the  $(*)$ -stable tuple  $\mathring{\underline{A}}$ . For*

$$h \in \overset{\circ}{\mathcal{H}}$$

$$\begin{aligned}\gamma \overset{\circ}{D}_* h &= \hat{D}_* h, \\ (\mathbf{1} \otimes \gamma) \overset{\circ}{C} h &= \hat{C} h,\end{aligned}$$

$$(\mathbf{1} \otimes \gamma) \theta_{\overset{\circ}{A}} d_h^i = \hat{\theta} d_h^i \text{ for } i \in \Lambda.$$

In other words, the part of  $\hat{\theta}$  described by case II in the previous section is equivalent to  $\theta_{\overset{\circ}{A}}$ .

*Proof.* We only have to use Lemma 3.5.1 and compare Proposition 3.4.2 and equation 3.13 as well as equations 3.12 and 3.14. For the latter note that  $d_h^i = \overset{\circ}{D}(0, \dots, 0, h, 0 \dots, 0)$ , where  $h$  is embedded at the  $i$ -th position. Hence

$$\begin{aligned}\gamma \sum_j \overset{\circ}{A}_j P_j d_h^i &= \gamma \sum_j \overset{\circ}{A}_j P_j \overset{\circ}{D}(0, \dots, 0, h, 0 \dots, 0) = \gamma \overset{\circ}{A} \overset{\circ}{D}(0, \dots, 0, h, 0 \dots, 0) \\ &= \gamma \overset{\circ}{D}_* \overset{\circ}{A}(0, \dots, 0, h, 0 \dots, 0) = \hat{D}_* \overset{\circ}{A} h\end{aligned}$$

and also

$$P_j \overset{\circ}{D} d_h^i = P_j \overset{\circ}{D}^2(0, \dots, 0, h, 0 \dots, 0) = (\delta_{ji} \mathbf{1} - \overset{\circ}{A}_j^* \overset{\circ}{A}_i) h.$$

□

Of course, Theorem 3.5.2 explains why we have called  $\hat{\theta}$  an *extended* characteristic function.

### 3.6 The extended characteristic function is a complete unitary invariant

In this section we prove that the extended characteristic function is a complete invariant with respect to unitary equivalence for the row contractions investigated in this paper. Suppose that  $\underline{A} = (A_1, \dots, A_d)$

and  $\underline{B} = (B_1, \dots, B_d)$  are ergodic and coisometric row contractions on Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$  such that  $A_i^* \Omega_A = \overline{\omega}_i \Omega_A$  and  $B_i^* \Omega_B = \overline{\omega}_i \Omega_B$  for  $i = 1, \dots, d$ , where  $\Omega_A \in \mathcal{H}_A$  and  $\Omega_B \in \mathcal{H}_B$  are unit vectors and  $\underline{\omega} = (\omega_1, \dots, \omega_d)$  is a tuple of complex numbers. Recall from Remark 3.2.2 that it is no serious restriction of generality to assume that it is the same tuple of complex numbers in both cases because this can always be achieved by a transformation with a unitary  $d \times d$ -matrix (with scalar entries). We will use all the notations introduced earlier with subscripts  $A$  or  $B$ .

Let us say that the extended characteristic functions  $\hat{\theta}_A$  and  $\hat{\theta}_B$  are *equivalent* if there exists a unitary  $V : \mathcal{D}_A \rightarrow \mathcal{D}_B$  such that  $\hat{\theta}_A = \hat{\theta}_B V$ . Note that the ranges of  $\hat{\theta}_A$  and  $\hat{\theta}_B$  are both contained in  $\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega$  and thus this definition makes sense. Let us further say that  $\underline{A}$  and  $\underline{B}$  are *unitarily equivalent* if there exists a unitary  $U : \mathcal{H}_A \rightarrow \mathcal{H}_B$  such that  $U A_i = B_i U$  for  $i = 1, \dots, d$ . By ergodicity the unit eigenvector  $\Omega_A$  (resp.  $\Omega_B$ ) is determined up to an unimodular constant (see Theorem 3.1.1(b)) and hence in the case of unitary equivalence we can always modify  $U$  to satisfy additionally  $U \Omega_A = \Omega_B$ .

**Theorem 3.6.1.** *The extended characteristic functions  $\hat{\theta}_A$  and  $\hat{\theta}_B$  are equivalent if and only if  $\underline{A}$  and  $\underline{B}$  are unitarily equivalent.*

*Proof.* If  $\underline{A}$  and  $\underline{B}$  are unitarily equivalent then all constructions differ only by naming and it follows that  $\hat{\theta}_A$  and  $\hat{\theta}_B$  are equivalent. Conversely, assume that there is a unitary  $V : \mathcal{D}_A \rightarrow \mathcal{D}_B$  such that  $\hat{\theta}_A = \hat{\theta}_B V$ . Now from the commuting diagram 3.5 and the definitions following it

$$\begin{aligned}
 W_B \mathcal{H}_B &= \mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega) \ominus M_{\hat{\theta}_B}(\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_B) \\
 &= \mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega) \ominus M_{\hat{\theta}_B}(\Gamma(\mathbb{C}^d) \otimes V \mathcal{D}_A) \\
 &= \mathbb{C} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega) \ominus M_{\hat{\theta}_A}(\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_A) \\
 &= W_A \mathcal{H}_A,
 \end{aligned}$$

where we used equation 3.8, i.e.,  $M_{\hat{\Theta}}(L_i \otimes \mathbf{1}_{\mathcal{D}}) = (L_i \otimes \mathbf{1}_{\mathcal{D}_\omega})M_{\hat{\Theta}}$ ,  $\forall 1 \leq i \leq d$ , to deduce  $M_{\hat{\Theta}_A} = M_{\hat{\Theta}_B}(\mathbf{1} \otimes V)$  from  $\hat{\theta}_A = \hat{\theta}_B V$ . Now we define the unitary  $U$  by

$$U := W_B^{-1} W_A|_{\mathcal{H}_A} : \mathcal{H}_A \rightarrow \mathcal{H}_B.$$

Because  $W_A \Omega_A = 1 = W_B \Omega_B$  we have  $U \Omega_A = \Omega_B$ . Further for all  $i = 1, \dots, d$  and  $h \in \mathcal{H}_A$ ,

$$\begin{aligned} U A_i h &= W_B^{-1} W_A A_i h = W_B^{-1} W_A P_{\mathcal{H}_A} V_i^A h = P_{\mathcal{H}_B} W_B^{-1} W_A V_i^A h \\ &= P_{\mathcal{H}_B} W_B^{-1} \tilde{V}_i W_A h = P_{\mathcal{H}_B} V_i^B W_B^{-1} W_A h = B_i U h, \end{aligned}$$

i.e.,  $\underline{A}$  and  $\underline{B}$  are unitarily equivalent.  $\square$

**Remark 3.6.2.** *An analogous result for completely non-coisometric tuples has been shown by G. Popescu in [Po89b], Theorem 3.5.4.*

Note further that if we change  $\underline{A} = (A_1, \dots, A_d)$  into  $\underline{A}' = (A'_1, \dots, A'_d)$  by applying a unitary  $d \times d$ -matrix with scalar entries (as described in Remark 3.2.2), then  $\hat{\theta}_A = \hat{\theta}_{A'}$ . In fact, this follows immediately from the definition of  $W$  as an intertwiner in Section 3.3, from which it is evident that  $W$  does not change if we take the same linear combinations on the left and on the right. This does not contradict Theorem 3.6.1 because  $\underline{\omega}$  and  $\underline{\omega}'$  are now different tuples of eigenvalues and Theorem 3.6.1 is only applicable when the same tuple of eigenvalues is used for  $\underline{A}$  and  $\underline{B}$ .

For another interpretation, let  $Z$  be a normal, unital, ergodic, completely positive map with an invariant vector state  $\langle \Omega_A, \cdot \Omega_A \rangle$ . If we consider two minimal Kraus decompositions of  $Z$ , i.e.,

$$Z = \sum_{i=1}^d A_i \cdot A_i^* = \sum_{i=1}^d A'_i \cdot (A'_i)^*,$$

with  $d$  minimal, then the tuples  $\underline{A} = (A_1, \dots, A_d)$  into  $\underline{A}' = (A'_1, \dots, A'_d)$  are related in the way considered above (see for example [Go04], A.2).

It follows that  $\hat{\theta}_A = \hat{\theta}_{A'}$  does not depend on the decomposition but can be associated to  $Z$  itself. Hence we have the following reformulation of Theorem 3.6.1.

**Corollary 3.6.3.** *Let  $Z_1, Z_2$  be normal, unital, ergodic, completely positive maps on  $B(\mathcal{H}_1), B(\mathcal{H}_2)$  with invariant vector states  $\langle \Omega_1, \cdot \Omega_1 \rangle$  and  $\langle \Omega_2, \cdot \Omega_2 \rangle$ . Then the associated extended characteristic functions  $\hat{\theta}_1$  and  $\hat{\theta}_2$  are equivalent if and only if  $Z_1$  and  $Z_2$  are conjugate, i.e., there exists a unitary  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that*

$$Z_1(x) = U^* Z_2(UxU^*)U \quad \text{for all } x \in B(\mathcal{H}_1).$$

### 3.7 Example

The following example illustrates some of the constructions in this paper.

Consider  $\mathcal{H} = \mathbb{C}^3$  and

$$A_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, A_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then  $\sum_{i=1}^2 A_i A_i^* = \mathbf{1}$ . Take the unital completely positive map  $Z : M_3 \rightarrow M_3$  by  $Z(x) = \sum_{i=1}^2 A_i x A_i^*$ . It is shown in Section 3.5 of [GKL06] (and not difficult to verify directly) that this map is ergodic. We will use the same notations here as in previous sections. Observe that the vector  $\Omega_{\mathcal{H}} := \frac{1}{\sqrt{3}}(1, 1, 1)^T$  gives an invariant vector state for  $Z$  as

$$\langle \Omega_{\mathcal{H}}, Z(x) \Omega_{\mathcal{H}} \rangle = \langle \Omega_{\mathcal{H}}, x \Omega_{\mathcal{H}} \rangle = \frac{1}{3} \sum_{i,j=1}^3 x_{ij}.$$



$A_i^* \Omega_{\mathcal{H}} = \frac{1}{\sqrt{2}} \Omega_{\mathcal{H}}$  and hence  $\underline{\omega} = \frac{1}{\sqrt{2}}(1, 1)$ . The orthogonal complement  $\mathring{\mathcal{H}}$  of  $\mathbb{C}\Omega_{\mathcal{H}}$  in  $\mathbb{C}^3$  and the orthogonal projection  $Q$  onto  $\mathring{\mathcal{H}}$  are given by

$$\mathring{\mathcal{H}} = \left\{ \begin{pmatrix} k_1 \\ k_2 \\ -(k_1 + k_2) \end{pmatrix} : k_1, k_2 \in \mathbb{C} \right\}, \quad Q = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

From this we get for  $\mathring{A}_i = Q A_i Q = A_i Q$

$$\mathring{A}_1 = \frac{1}{3\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & -1 \\ -2 & 1 & 1 \end{pmatrix}, \quad \mathring{A}_2 = \frac{1}{3\sqrt{2}} \begin{pmatrix} 1 & 1 & -2 \\ -1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We notice that the tuple  $\mathring{\underline{A}} = (\mathring{A}_1, \mathring{A}_2)$  is  $*$ -stable as (by induction)

$$\sum_{|\alpha|=n} \mathring{A}_\alpha \mathring{A}_\alpha^* = \frac{1}{3 \times 2^{n-1}} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow 0 \quad (n \rightarrow \infty).$$

Here  $\mathcal{P} = \mathbb{C}^2$  and  $\mathring{\mathcal{P}} := \mathcal{P} \ominus \mathbb{C}\Omega_{\mathcal{P}}$  with  $\Omega_{\mathcal{P}} = \frac{1}{\sqrt{2}}(1, 1)^T$ . Easy calculation shows that  $\hat{D}_* : \mathring{\mathcal{H}} \rightarrow \mathring{\mathcal{P}}$  is given by

$$\begin{pmatrix} k_1 \\ k_2 \\ -(k_1 + k_2) \end{pmatrix} \mapsto \frac{1}{\sqrt{6}}(2k_1 + k_2) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Moreover  $\mathring{D}_* = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ . There exists an isometry  $\gamma : \mathring{\mathcal{D}}_* \rightarrow \mathring{\mathcal{P}}$

such that  $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 1 \end{pmatrix}$  and  $\gamma(\mathring{D}_* h) = \hat{D}_* h$  for  $h \in \mathring{\mathcal{H}}$ .

The map  $\hat{C} : \mathcal{H} \rightarrow \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega$  is given by  $\hat{C}(\Omega_{\mathcal{H}}) = 1$  and for  $h \in \mathring{\mathcal{H}}$  by

$$\begin{aligned} & \hat{C} \begin{pmatrix} k_1 \\ k_2 \\ -(k_1 + k_2) \end{pmatrix} \\ &= e_0 \otimes \frac{(2k_1 + k_2)}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{\alpha, \alpha_1=1} e_\alpha \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{(k_1 + 2k_2)}{\sqrt{6}} \\ & \quad \times \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{\alpha, \alpha_1=2} e_\alpha \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{(k_1 - k_2)}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned}$$

where the summations are taken over all  $0 \neq \alpha \in \tilde{\Lambda}$  such that  $\alpha_i \neq \alpha_{i+1}$  for all  $1 \leq i \leq |\alpha|$  and fixing  $\alpha_1$  to 1 or 2 as indicated. This simplification occurs because  $\mathring{A}_i^2 = 0$  for  $i = 1, 2$ . All the summations below in this section are also of the same kind.

Now using the equations 3.11 and 3.12 for  $\hat{\theta}_A : \mathcal{D}_A \rightarrow \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_\omega$  and simplifying we get

$$\begin{aligned} \hat{\theta}_A d_{\Omega_{\mathcal{H}}}^1 &= -e_0 \otimes \frac{1}{6} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{\alpha, \alpha_1=1} e_\alpha \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{1}{6} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ & \quad + \sum_{\alpha, \alpha_1=2} e_\alpha \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{1}{6} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\hat{\theta}_A d_{\Omega_{\mathcal{H}}}^2, \end{aligned}$$

and for  $h \in \mathring{\mathcal{H}}$ ,

$$\begin{aligned} \hat{\theta}_A d_h^1 &= -e_0 \otimes \frac{k_1}{2\sqrt{3}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e_1 \otimes \frac{(k_1 + k_2)}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{\alpha, \alpha_1=1} e_1 \otimes e_\alpha \\ & \quad \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{(k_1 + 2k_2)}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sum_{\alpha, \alpha_1=2} e_1 \otimes e_\alpha \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{k_2}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ & \quad + \sum_{\alpha, \alpha_1=2} e_\alpha \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{k_1}{2\sqrt{3}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
\hat{\theta}_A d_h^2 &= -e_0 \otimes \frac{(k_1 + k_2)}{2\sqrt{3}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{\alpha, \alpha_1=1} e_\alpha \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{(k_1 + k_2)}{2\sqrt{3}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&\quad + e_2 \otimes \frac{k_1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sum_{\alpha, \alpha_1=1} e_2 \otimes e_\alpha \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{k_2}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&\quad + \sum_{\alpha, \alpha_1=2} e_2 \otimes e_\alpha \otimes \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} \frac{(k_1 - k_2)}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\end{aligned}$$

Form this we can easily obtain  $\overset{\circ}{C}$  and  $\theta_A^\circ$  for  $h \in \overset{\circ}{\mathcal{H}}$  by using the following relations from Theorem 3.5.2,

$$(\mathbf{1} \otimes \gamma) \overset{\circ}{C} h = \hat{C} h,$$

$$(\mathbf{1} \otimes \gamma) \theta_A^\circ d_h^i = \hat{\theta}_A d_h^i.$$

Further

$$l_1 = A_1 \Omega_{\mathcal{H}} - \frac{1}{\sqrt{2}} \Omega_{\mathcal{H}} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad l_2 = A_2 \Omega_{\mathcal{H}} - \frac{1}{\sqrt{2}} \Omega_{\mathcal{H}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\mathring{A}_1 l_1 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ and clearly } \overset{\circ}{\mathcal{H}} = \overline{\text{span}}\{\mathring{A}_\alpha l_i : i = 1, 2 \text{ and } \alpha \in \tilde{\Lambda}\}, \text{ as}$$

already observed in Remark 3.1.

## 3.8 Appendix

Here for  $\ast$ -stable  $\underline{T}$  we generalize the computation of [Go06] to tuples. The following result (together with the results in Section 3.1) made us to

ask and investigate the relation between Popescu's characteristic function and ergodic tuples. Let

$$R_k : \mathcal{H} \oplus (\oplus_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes \tilde{\mathcal{D}}_\alpha) \rightarrow \mathcal{H} \oplus (\oplus_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes \mathcal{D}) = \mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D})$$

$$\text{where } \tilde{\mathcal{D}}_\alpha = \begin{cases} \mathcal{D}_* & \text{if } |\alpha| = k \\ \mathcal{D} & \text{if } |\alpha| \neq k \end{cases} \text{ be given by}$$

$$\begin{aligned} h \oplus \left( \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes \tilde{d}_\alpha \right) &\mapsto \left\{ \sum_i T_i h + D_* \left( \sum_{|\alpha|=k} \tilde{d}_\alpha \right) \right\} \oplus \left[ \sum_{|\alpha|<k} e_\alpha \otimes \tilde{d}_\alpha + \sum_{|\alpha|=k} e_\alpha \right. \\ &\quad \left. \otimes \{D(h, \dots, h) - (T_1^* \tilde{d}_\alpha, \dots, T_d^* \tilde{d}_\alpha)\} + \sum_{|\alpha|>k} e_\alpha \otimes \tilde{d}_\alpha \right]. \end{aligned}$$

Let us use the presentation of the minimal isometric dilation given by Popescu on  $\mathcal{H} \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D})$ . First consider the isometry  $U := \frac{1}{\sqrt{d}} \sum_{i=1}^d V_i$ .

$$\begin{aligned} &d^{\frac{k}{2}} U^k (h \oplus \sum_{\delta \in \tilde{\Lambda}} e_\delta \otimes d_\delta) \\ &= \sum_{|\alpha|=k} T_\alpha h \oplus \sum_{i=0}^{k-1} \sum_{|\beta|=i} e_\beta \otimes D \left( \sum_{|\gamma|=k-1-i} T_\gamma h, \dots, \sum_{|\gamma|=k-1-i} T_\gamma h \right) \\ &\quad + \sum_{|\epsilon|=k} e_\epsilon \otimes \sum_{\delta \in \tilde{\Lambda}} e_\delta \otimes d_\delta = R_0 \dots R_{k-1} (h \oplus \sum_{|\epsilon|=k} e_\epsilon \otimes \sum_{\delta \in \tilde{\Lambda}} e_\delta \otimes d_\delta). \end{aligned}$$

We conclude that  $U^k = R_0 \dots R_{k-1} (\mathbf{1}_{\mathcal{H}} \oplus (L \otimes \mathbf{1}))^k$  with  $L := \frac{1}{\sqrt{d}} \sum_{i=1}^d L_i$ , i.e., the product of  $R_i$ 's is a kind of cocycle relating the isometries  $U$  and  $L$ .

On the other hand we can use the product of adjoints to factorize the unitary  $\hat{W}$  corresponding to the characteristic function. Note that

$$\begin{aligned} R_k^* (h \oplus \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha) &= \left( \sum_i T_i^* h + \sum_i \sum_{|\alpha|=k} P_i D d_\alpha \right) \oplus \left( \sum_{|\alpha|<k} e_\alpha \otimes d_\alpha \right. \\ &\quad \left. + \sum_{|\alpha|=k} e_\alpha \otimes (D_* h - \sum_i T_i P_i d_\alpha) + \sum_{|\alpha|>k} e_\alpha \otimes d_\alpha \right), \end{aligned}$$

where  $P_i$  is the projection onto the  $i^{th}$  component of  $\mathcal{D}$ .

$$\begin{aligned}
& R_{k-1}^* \dots R_0^* (h \oplus \sum_{\alpha \in \bar{\Lambda}} e_\alpha \otimes d_\alpha) \\
= & \left\{ \sum_{|\beta|=k} T_\beta^* h + \sum_{|\beta|=k-1} \sum_i T_\beta^* P_i D d_0 + \dots + \sum_i \sum_{|\beta|=k-1} P_i D d_\beta \right\} \\
& \oplus [(D_* h - \sum_i T_i P_i d_0) \\
& + \sum_{|\beta|=1} e_\beta \otimes \{ \sum_{|\gamma|=1} D_* T_\gamma^* h + \sum_i D_* P_i D d_0 - \sum_{|\gamma|=1} \sum_i T_i P_i d_\gamma \} \\
& + \dots + \sum_{|\beta|=k-1} e_\beta \otimes \{ \sum_{|\gamma|=k-1} D_* T_\gamma^* h + \sum_{|\gamma|=k-2} D_* \sum_i T_\gamma^* P_i D d_0 + \dots \\
& - \sum_{|\gamma|=d-1} \sum_i T_i P_i d_\gamma \} + \sum_{|\alpha|>k-1} e_\alpha \otimes d_\alpha].
\end{aligned}$$

Now the first bracket  $\{\cdot\}$  converges to 0 for  $k \rightarrow \infty$ , and a comparison of the second bracket  $[\cdot]$  with Proposition 3.3.1 shows that

$$\hat{W} = \lim_{k \rightarrow \infty} R_{k-1}^* \dots R_0^*,$$

which is analogous to the product formula for  $\tilde{\mathbf{w}}$  in Theorem 3.1.1.



## Chapter 4

# Characteristic Functions of Liftings

**Abstract:** *We introduce characteristic functions for certain contractive liftings of row contractions. These are multi-analytic operators which classify the liftings up to unitary equivalence and provide a kind of functional model. The most important cases are subisometric and coisometric liftings. We also identify the most general setting which we call reduced liftings. We derive properties of these new characteristic functions and discuss the relation to Popescu's definition of the characteristic function for completely non-coisometric row contractions. Finally we apply our theory to completely positive maps and prove a one-to-one correspondence between the fixed point sets of completely positive maps related to each other by a subisometric lifting.*

Joint work with Rolf Gohm, accepted in Journal of Operator Theory.

## Introduction

Let  $C$  be a contraction on a Hilbert space  $\mathcal{H}_C$ . Then a contraction  $E$  on a Hilbert space  $\mathcal{H}_E \supset \mathcal{H}_C$  is called a contractive lifting of  $C$  if  $PE = CP$ , where  $P$  is the orthogonal projection from  $\mathcal{H}_E$  onto  $\mathcal{H}_C$ . In other words, we have an operator matrix

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}. \quad (4.1)$$

See Chapter 5 of [FF90]. In this book C.Foias and A.E.Frazho amply demonstrate the importance of understanding the structure of contractive liftings, in particular in connection with the commutant lifting theorem and its applications.

The minimal isometric dilation (mid for short) of  $C$  is the most prominent example of a contractive lifting. In [DF84] R.G. Douglas and C. Foias introduced subisometric dilations (see also Chapter 8.3 of [Ber88] for a discussion closer to our point of view). These are contractive liftings with the property that the mid of  $E$  is also minimal as an isometric dilation of  $C$ . In this context Douglas and Foias were especially interested in problems of uniqueness and of commutant lifting. We arrived at the subisometric property in a completely different way and ask different questions about it. Let us briefly describe the most relevant aspects of this development.

Many results of the Sz.-Nagy/Foias-theory for contractions [NF70] can be generalized to row contractions  $\underline{C} = (C_1, \dots, C_d)$ , i.e. tuples of operators such that  $\sum_{i=1}^d C_i C_i^* \leq \mathbf{1}$ . This has been done very systematically by G.Popescu starting with [Po89a] and many people contributed to this development, an incomplete list of work related to our interests is [Ar98, BBD04, BDZ06, BES05, DKS01, Po89b, Po03, Po05]. In particular in [Po89b] G.Popescu described a class of multi-analytic operators which classify completely non-coisometric (c.n.c.) row contractions up to



unitary equivalence and called them characteristic functions, in analogy to a similar concept in the Sz.-Nagy/Foias-theory. In [DG07a] S. Dey and R. Gohm started from some seemingly unrelated questions in noncommutative probability theory arising in [Go04, GKL06] and established a class of multi-analytic operators which are associated to certain rather special coisometric row contractions (i.e.,  $\sum_{i=1}^d C_i C_i^* = \mathbf{1}$ ). Investigating their properties we came to the conclusion that there are good reasons to think of them as of characteristic functions for these tuples. This is not covered by Popescu's theory.

In this paper we will show that it is the property of being a subisometric lifting which makes this analysis possible. This is a vast generalization of the setting of [DG07a] and it clarifies the mechanism behind it. It is straightforward to define liftings for row contractions. Let  $\underline{E} = (E_1, \dots, E_d)$  be a row contraction on a Hilbert space  $\mathcal{H}_E \supset \mathcal{H}_C$ . If for all  $i = 1, \dots, d$  (with  $d$  countable) we have an operator matrix

$$E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix} \quad (4.2)$$

with respect to  $\mathcal{H}_C \oplus \mathcal{H}_C^\perp$  then we say that  $\underline{E}$  is a lifting of  $\underline{C} = (C_1, \dots, C_d)$  by  $\underline{A} = (A_1, \dots, A_d)$  (or that  $\underline{E}$  is an extension of  $\underline{A}$  by  $\underline{C}$ ). The subisometric property in the form given here also makes sense for row contractions, using Popescu's theory of mid for row contractions [Po89a]. This is worked out in Section 4.1 below. It then turns out that there is a Beurling-type classification of subisometric liftings, involving a correspondence to certain multi-analytic inner operators (Theorem 4.1.6). They classify subisometric liftings up to unitary equivalence, so we call them characteristic functions of (subisometric) liftings.

In Section 4.2 we focus on coisometric liftings, i.e.  $\sum_{i=1}^d E_i E_i^* = \mathbf{1}$ , emphasizing another type of classification which uses an isometry  $\gamma$  mapping the defect space  $\mathcal{D}_{*,A}$  of  $\underline{A}$  into the defect space  $\mathcal{D}_C$  of  $\underline{C}$  (Theorem

4.2.1). The connection to Section 4.1 lies in the fact that coisometric liftings by  $*$ -stable  $\underline{A}$  are subisometric (Proposition 4.2.3). But this is only a special case and we have to generalize further.

This is done in Section 4.3. We get a hint from a result about contractive liftings for single contractions. Lemma 2.1 in Chap.IV of [FF90] states that  $E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$  is a contraction if and only if  $C$  and  $A$  are contractions and there exists a contraction  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  such that

$$B = D_{*,A} \gamma^* D_C, \quad (4.3)$$

where  $D_{*,A}$  and  $D_C$  are the defect operators of  $A^*$  and  $C$ . We establish an analogous result for row contractions (Proposition 4.3.1). This shows that the isometry  $\gamma$  occurring for coisometric liftings in Section 4.2 has to be replaced in a more general setting by a contraction.

The most general situation where we can establish a satisfactory theory of characteristic functions for liftings is identified in Section 4.3 and we call such liftings reduced. The technical tool here is to use the Wold decomposition for the mid's. For  $\gamma$  we isolate the special property needed and call it resolving. Reduced liftings include subisometric liftings as well as coisometric liftings by c.n.c. row contractions. We define characteristic functions for reduced liftings (Definition 4.3.6) and we argue that this is the most general setting which is natural for that. These characteristic functions are multi-analytic operators (not inner in general) and they characterize reduced liftings up to unitary equivalence. They also provide a kind of functional model for the lifting which is useful for a closer investigation of the structure of the lifting in the same sense as the characteristic functions of Sz.-Nagy/Foias and of Popescu are useful in their context.

In Section 4.4 we study some further properties of these characteristic functions. In particular we clarify the connection to Popescu's characteristic functions and we investigate iterated liftings, showing a factorization

result for our characteristic functions (Theorem 4.4.1). This is another indication that our definition leads to a promising theory.

We believe that in particular the theory of subisometric liftings may be even more interesting for row contractions than it is for single contractions. There is a straightforward way to transfer results from a row contraction  $\underline{C} = (C_1, \dots, C_d)$  to the completely positive map  $\Phi_C : X \mapsto \sum_{i=1}^d C_i X C_i^*$ . This topic is taken up in Section 4.5. We define characteristic functions for liftings of completely positive maps and show in which way they are characteristic in this case (Corollary 4.5.2). We investigate what subisometric lifting means in this context and prove a one-to-one correspondence between the fixed point sets (Theorem 4.5.4). In particular we consider the situation where a normal invariant state is restricted to its support (Corollary 4.5.6). From our point of view these applications give a strong motivation for further developing the theory of liftings for row contractions.

In an Appendix we reprove a commutant lifting theorem by O. Bratteli, P. Jorgensen, A. Kishimoto and R.F. Werner [BJKW00], used in Section 4.5, in a way that helps to understand its role in our theory.

## 4.1 Subisometric liftings

In this section we define subisometric liftings in the setting of row contractions and show that there is a nice Beurling-type classification for them.

We recall the notion of a minimal isometric dilation for a row contraction, cf. [Po89a]. Let  $\underline{T} = (T_1, \dots, T_d)$  be a row contraction on a Hilbert space  $\mathcal{H}$ . Treating  $\underline{T}$  as an operator from  $\bigoplus_{i=1}^d \mathcal{H}$  to  $\mathcal{H}$ , define  $D_* := (\mathbf{1} - \underline{T}\underline{T}^*)^{\frac{1}{2}} : \mathcal{H} \rightarrow \mathcal{H}$  and  $D := (\mathbf{1} - \underline{T}^*\underline{T})^{\frac{1}{2}} : \bigoplus_{i=1}^d \mathcal{H} \rightarrow \bigoplus_{i=1}^d \mathcal{H}$ .

This implies that

$$D_* = (\mathbf{1} - \sum_{i=1}^d T_i T_i^*)^{\frac{1}{2}}, \quad D = (\delta_{ij} \mathbf{1} - T_i^* T_j)^{\frac{1}{2}}_{d \times d} \quad (4.4)$$

Let  $\mathcal{D}_* := \overline{\text{range } D_*}$  and  $\mathcal{D} := \overline{\text{range } D}$ .

We use the following *multi-index notation*. Let  $\Lambda$  denote the set  $\{1, 2, \dots, d\}$  and  $\tilde{\Lambda} := \cup_{n=0}^{\infty} \Lambda^n$ , where  $\Lambda^0 := \{0\}$ . If  $\alpha \in \Lambda^n \subset \tilde{\Lambda}$  the integer  $n = |\alpha|$  is called its length. Now  $T_\alpha$  with  $\alpha = (\alpha_1, \dots, \alpha_n) \in \Lambda^n$  means  $T_{\alpha_1} T_{\alpha_2} \dots T_{\alpha_n}$ .

The full Fock space over  $\mathbb{C}^d$  ( $d \geq 2$ ) denoted by  $\Gamma(\mathbb{C}^d)$  is

$$\Gamma(\mathbb{C}^d) := \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes 2} \oplus \dots \oplus (\mathbb{C}^d)^{\otimes m} \oplus \dots \quad (4.5)$$

To simplify notation we shall often only write  $\Gamma$  instead of  $\Gamma(\mathbb{C}^d)$ . The vector  $e_0 := 1 \oplus 0 \oplus \dots$  is called the vacuum vector. Let  $e_1, \dots, e_d$  be the standard orthonormal basis of  $\mathbb{C}^d$ . We include  $d = \infty$  in which case  $\mathbb{C}^d$  stands for a complex separable Hilbert space of infinite dimension. For  $\alpha \in \Lambda^n$ ,  $e_\alpha$  will denote the vector  $e_{\alpha_1} \otimes e_{\alpha_2} \otimes \dots \otimes e_{\alpha_n}$  in the full Fock space  $\Gamma$ . Then  $e_\alpha$  over all  $\alpha \in \tilde{\Lambda}$  forms an orthonormal basis of the full Fock space. The (left) creation operators  $L_i$  on  $\Gamma(\mathbb{C}^d)$  are defined by  $L_i x = e_i \otimes x$  for  $1 \leq i \leq d$  and  $x \in \Gamma(\mathbb{C}^d)$ . Then  $\underline{L} = (L_1, \dots, L_d)$  is a row isometry, i.e., the  $L_i$  are isometries with orthogonal ranges.

Using the definition of lifting in the introduction a minimal isometric dilation (mid for short) can be described as an isometric lifting  $\underline{V}$  of  $\underline{T}$  such that the spaces  $V_\alpha \mathcal{H}$  with  $\alpha \in \tilde{\Lambda}$  together span the Hilbert space on which the  $V_i$  are defined. It is an important fact, which we shall use repeatedly, that such minimal isometric dilations are unique up to unitary equivalence (cf. [Po89a]). A useful model for the mid is given by a version of the Schäffer construction, given in [Po89a]. Namely, we can realize a

mid  $\underline{V}$  of  $\underline{T}$  on the Hilbert space  $\hat{\mathcal{H}} := \mathcal{H} \oplus (\Gamma \otimes \mathcal{D})$ ,

$$V_i(h \oplus \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha) = T_i h \oplus [e_0 \otimes D_i h + e_i \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha] \quad (4.6)$$

for  $h \in \mathcal{H}$  and  $d_\alpha \in \mathcal{D}$ . Here  $D_i h := D(0, \dots, 0, h, 0, \dots, 0)$  and  $h$  is embedded at the  $i^{\text{th}}$  component.

If we have more than one row contraction at the same time then we shall use the above notations with superscripts or subscripts, as convenient. We are now ready for the basic definition in this section.

**Definition 4.1.1.** Let  $\underline{C} = (C_1, \dots, C_d)$  be a row contraction on a Hilbert space  $\mathcal{H}_C$ . A lifting  $\underline{E}$  of  $\underline{C}$  on  $\mathcal{H}_E \supset \mathcal{H}_C$  is called *subisometric* if the corresponding mids  $\underline{V}^E$  (on the Hilbert space  $\hat{\mathcal{H}}_E$ ) and  $\underline{V}^C$  (on the Hilbert space  $\hat{\mathcal{H}}_C$ ) are unitarily equivalent, in the sense that there exists a unitary  $W : \hat{\mathcal{H}}_E \rightarrow \hat{\mathcal{H}}_C$  such that  $W|_{\mathcal{H}_C} = \mathbf{1}|_{\mathcal{H}_C}$  and  $WV_i^E = V_i^C W$ .

For  $d = 1$  this is consistent with the definition of subisometric dilation in [DF84], see the discussion in the introduction. Note that the mid  $\underline{V}^C$  is an example of a subisometric lifting in this sense. Another (trivial) example is  $\underline{C}$  itself (considered as a lifting of  $\underline{C}$ ). Further note that, given the mids  $\underline{V}^E$  and  $\underline{V}^C$ , the unitary  $W$  is uniquely determined by its properties (use the minimality of  $\underline{V}^C$ ).

We want to make the structure of subisometric liftings more explicit. Let  $\underline{E} = (E_1, \dots, E_d)$  be a subisometric lifting of  $\underline{C} = (C_1, \dots, C_d)$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  as in Definition 4.1.1, so that for all  $i = 1, \dots, d$  we have block matrices

$$E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix} \quad (4.7)$$

Let  $\underline{V}^C$  be the mid of  $\underline{C}$ , realized as in (4.6) on the space  $\hat{\mathcal{H}}_C = \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C)$ . Because  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A \subset \hat{\mathcal{H}}_E$  we can use the unitary  $W$  from the

subisometric lifting property to obtain a subspace  $\mathcal{H}_{A*} := W\mathcal{H}_A \subset \Gamma \otimes \mathcal{D}_C$ . Further  $\mathcal{H}_{E*} := \mathcal{H}_C \oplus \mathcal{H}_{A*} \subset \hat{\mathcal{H}}_C$ , and  $\underline{V}^C$  is also a mid of the row contraction  $\underline{E}_*$  which is transferred by  $W$  from the unitarily equivalent original  $\underline{E}$ . We can write

$$E_{i*} = \begin{pmatrix} C_i & 0 \\ B_{i*} & A_{i*} \end{pmatrix} \quad (4.8)$$

so  $\underline{E}_*$  is also a lifting of  $\underline{C}$ .

Because  $\underline{V}^C$  is a mid of  $\underline{E}_*$  it follows that  $\mathcal{H}_{E*}$  is coinvariant for  $\underline{V}^C$  (by which we mean that it is invariant for all  $(V_i^C)^*$ ,  $i = 1, \dots, d$ ). Note that

$$V_i^C|_{\Gamma \otimes \mathcal{D}_C} = L_i \otimes \mathbf{1}. \quad (4.9)$$

Hence  $\underline{L} \otimes \mathbf{1}$  is an isometric lifting of  $\underline{A}_*$ , in particular  $\mathcal{H}_{A*}$  is coinvariant for  $\underline{L} \otimes \mathbf{1}$ . An isometric lifting always contains the mid. In particular the mid of  $\underline{A}_*$  lives on the space  $\overline{\text{span}}\{(L_\alpha \otimes \mathbf{1})\mathcal{H}_{A*}, \alpha \in \tilde{\Lambda}\}$ . This subspace is reducing for the  $L_i \otimes \mathbf{1}$  for all  $i = 1, \dots, d$  and hence has the form  $\Gamma \otimes \mathcal{E}$  for a subspace  $\mathcal{E}$  of  $\mathcal{D}_C$ , see for example Cor.1.7 of [Po05], where it is done in a more general setting. In this reference the space  $\mathcal{E}$  is described as the closure of the image of  $\mathcal{H}_{A*}$  under the orthogonal projection onto  $e_0 \otimes \mathcal{D}_C$ .

We can obtain a more concrete formula for  $\mathcal{E}$  by comparing this result with another way of writing the mid. First note that, as a compression of  $\underline{L} \otimes \mathbf{1}$ , the row contraction  $\underline{A}_*$  (and hence also  $\underline{A}$ ) is  $*$ -stable, i.e., for all  $h \in \mathcal{H}_A$

$$\lim_{n \rightarrow \infty} \sum_{|\alpha|=n} \|A_\alpha^* h\|^2 = 0, \quad (4.10)$$

cf. [Po89a], Prop.2.3 (where it is called pure). In this case, with  $D_{*,A} =$

$(\mathbf{1} - \underline{AA}^*)^{\frac{1}{2}} : \mathcal{H}_A \rightarrow \mathcal{H}_A$  and  $\mathcal{D}_{*,A}$  its closed range, the map

$$\begin{aligned} \mathcal{H}_A &\rightarrow \Gamma \otimes \mathcal{D}_{*,A} \\ h &\mapsto \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes D_{*,A} A_\alpha^* h \end{aligned} \quad (4.11)$$

is isometric (Popescu's Poisson kernel, cf. [Po03]). With this embedding of  $\mathcal{H}_A$  it can be checked that now  $\underline{L} \otimes \underline{\mathbf{1}}$  on  $\Gamma \otimes \mathcal{D}_{*,A}$  is a mid of  $\underline{A}$ .

Because mids are unique up to unitary equivalence we have a unitary  $u : \Gamma \otimes \mathcal{D}_{*,A} \rightarrow \Gamma \otimes \mathcal{E}$  such that  $u\mathcal{H}_A = \mathcal{H}_{A*}$  and  $u(L_i \otimes \mathbf{1}) = (L_i \otimes \mathbf{1})u$  for all  $i = 1, \dots, d$ . The commutation relation implies that  $u$  is of the form  $\mathbf{1} \otimes u'$ , where  $u'$  is a unitary from  $\mathcal{D}_{*,A}$  onto  $\mathcal{E}$  (you may use the fact that  $e_0 \otimes \mathcal{D}_{*,A}$  respectively  $e_0 \otimes \mathcal{E}$  are the uniquely determined wandering subspaces). Thinking of  $u'$  as an isometry from  $\mathcal{D}_{*,A}$  into  $\mathcal{D}_C$  we call it  $\gamma$ . So  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  has  $\mathcal{E}$  as its range and it is canonically associated to a subisometric lifting in the way shown above.

Using  $\gamma$  we see that the embedding of  $\mathcal{H}_A$  into  $\Gamma \otimes \mathcal{D}_C$  is automatically of Poisson kernel type (4.11), namely

$$\mathcal{H}_A \ni h \mapsto \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes \gamma D_{*,A} A_\alpha^* h \in \Gamma \otimes \mathcal{D}_C \quad (4.12)$$

which is an explicit formula for the embedding  $W|_{\mathcal{H}_A} : \mathcal{H}_A \rightarrow \mathcal{H}_{A*} \subset \Gamma \otimes \mathcal{D}_C$ .

Note also that the isometry  $\gamma$  is closely related to the  $\underline{B}$ -part of the lifting  $\underline{E}$ . In fact, because  $E_{i*}^* = (V_i^C)^*|_{\mathcal{H}_{E*}}$  we obtain  $B_{i*}^* = p_C(V_i^C)^* p_{A*}$ , where  $p_C, p_{A*}$  are the orthogonal projections onto  $\mathcal{H}_C, \mathcal{H}_{A*}$ . Combining this with (4.6) and (4.12) yields  $B_i^* = D_{i,C}^* \gamma D_{*,A} : \mathcal{H}_A \rightarrow \mathcal{H}_C$ ,  $i = 1, \dots, d$ . Or in a more compact form

$$\underline{B}^* = D_C^* \gamma D_{*,A}. \quad (4.13)$$

**Proposition 4.1.2.** *A lifting  $\underline{E}$  of a row contraction  $\underline{C}$  with*

$$E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}, \quad i = 1, \dots, d,$$

*is subisometric if and only if  $\underline{A}$  is  $*$ -stable and  $\underline{B} = D_{*,A}\gamma^*D_C$  with an isometry  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$ .*

*Proof.* We have already seen above that if  $\underline{E}$  is subisometric then the conditions are satisfied. Conversely, if  $\underline{A}$  is  $*$ -stable then use the isometry  $\gamma$  to embed  $\underline{A}$  (as  $\underline{A}_*$ ) and its mid into  $\Gamma \otimes \mathcal{D}_C$  as in (4.12). Then the formula for  $\underline{B}$  (or (4.13)) combined with (4.6) for  $\underline{C}$  shows that  $\underline{V}^C$  is a mid for  $\underline{E}_*$  which is unitarily equivalent to  $\underline{E}$ . (Clearly  $\underline{V}^C$  is minimal for  $\underline{E}_*$  because it is already minimal for  $\underline{C}$ .) Hence  $\underline{E}$  is subisometric.  $\square$

**Remark 4.1.3.** *This is consistent with the results for  $d = 1$  in [DF84] which we mentioned in the introduction.  $\gamma$  unitary corresponds to what Douglas and Foias call a minimal subisometric dilation. We have no reason for imposing this condition and continue to consider general subisometric liftings. Compare also Chapter 8.3 of [Ber88].*

Classifying subisometric liftings becomes especially transparent by focusing on the invariant subspace associated to it.

**Definition 4.1.4.** *Let  $\underline{E}$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  be a subisometric lifting of  $\underline{C}$  on  $\mathcal{H}_C$ , notation as in Definition 4.1.1. Then we call*

$$\mathcal{N} := (\Gamma \otimes \mathcal{D}_C) \ominus W\mathcal{H}_A \tag{4.14}$$

*the invariant subspace associated to the subisometric lifting. Clearly  $\mathcal{N}$  is invariant for  $L_i \otimes \mathbf{1}$ ,  $i = 1, \dots, d$ .*

We can go the way back. Let  $\underline{C}$  on  $\mathcal{H}_C$  be a row contraction. If  $\mathcal{N} \subset \Gamma \otimes \mathcal{D}_C$  is a subspace which is invariant for all  $L_i \otimes \mathbf{1}$ ,  $i = 1, \dots, d$



then we can define

$$\mathcal{H}_{A*} := (\Gamma \otimes \mathcal{D}_C) \ominus \mathcal{N} \quad (4.15)$$

$$\mathcal{H}_* := \mathcal{H}_C \oplus \mathcal{H}_{A*} \quad (4.16)$$

On  $\mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C)$  we have the mid  $\underline{V}^C$  of  $\underline{C}$ , as in (4.6), so we can further define

$$\underline{E}_* = (E_{1*}, \dots, E_{d*}), \quad E_{i*} := P_{\mathcal{H}_*} V_i^C|_{\mathcal{H}_{E*}} : \mathcal{H}_{E*} \rightarrow \mathcal{H}_{E*} \quad (4.17)$$

Then  $\underline{E}_*$  is a row contraction and

$$E_{i*} = \begin{pmatrix} C_i & 0 \\ B_{i*} & A_{i*} \end{pmatrix} \quad (4.18)$$

with respect to the decomposition  $\mathcal{H}_{E*} := \mathcal{H}_C \oplus \mathcal{H}_{A*}$ , i.e.,  $\underline{E}_*$  is a lifting of  $\underline{C}$ . Then  $\underline{V}^C$  is a mid of  $\underline{E}_*$  (minimal because it is already minimal for  $\underline{C}$ ). Hence we have constructed a subisometric lifting. We are back in the setting of Proposition 4.1.2.

These considerations suggest a classification of subisometric liftings along a Beurling type theorem for the associated invariant subspaces. It is instructive to introduce the generalized inner functions occurring here directly from the definition of subisometric lifting.

So let  $\underline{E}$  be a subisometric lifting of  $\underline{C}$ . Then the mids  $\underline{V}^E$  of  $\underline{E}$  and  $\underline{V}^C$  of  $\underline{C}$  are connected by the unitary

$$W : \hat{\mathcal{H}}_E = \mathcal{H}_E \oplus (\Gamma \otimes \mathcal{D}_E) \rightarrow \hat{\mathcal{H}}_C = \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C) \quad (4.19)$$

such that  $W|_{\mathcal{H}_C} = \mathbf{1}|_{\mathcal{H}_C}$  and  $WV_i^E = V_i^C W$  for  $i = 1, \dots, d$ . If we define the isometry

$$M_{C,E} := W|_{\Gamma \otimes \mathcal{D}_E} \quad (4.20)$$

then from (4.6) and (4.19) we obtain

$$M_{C,E}(L_i \otimes \mathbf{1}_E) = (L_i \otimes \mathbf{1}_C)M_{C,E} \quad (4.21)$$

which means that  $M_{C,E} : \Gamma \otimes \mathcal{D}_E \rightarrow \Gamma \otimes \mathcal{D}_C$  is a *multi-analytic inner operator* determined by its *symbol*

$$\Theta_{C,E} : \mathcal{D}_E \rightarrow \Gamma \otimes \mathcal{D}_C, \quad \Theta_{C,E} = W|_{e_0 \otimes \mathcal{D}_E}. \quad (4.22)$$

according to the terminology introduced in [Po89b]. Obviously this is nothing but the multi-analytic inner operator corresponding to the invariant subspace  $\mathcal{N}$ , in fact it is easy to check that

$$\mathcal{N} = M_{C,E}(\Gamma \otimes \mathcal{D}_E), \quad (4.23)$$

compare the Beurling type theorem in [Po89b]. Our new insight is that it is connected to the subisometric lifting  $\underline{E}$  of  $\underline{C}$ .

**Definition 4.1.5.** *We call  $M_{C,E}$  (or  $\Theta_{C,E}$ ) the characteristic function of the subisometric lifting  $\underline{E}$  of  $\underline{C}$ .*

It is not difficult to check that two multi-analytic inner operators  $M : \Gamma \otimes \mathcal{D} \rightarrow \Gamma \otimes \mathcal{E}$  and  $M' : \Gamma \otimes \mathcal{D}' \rightarrow \Gamma \otimes \mathcal{E}$  with symbols  $\Theta, \Theta'$  describe the same invariant subspace if and only if there exists a unitary  $v : \mathcal{D} \rightarrow \mathcal{D}'$  such that  $\Theta = \Theta'v$ . Let us call multi-analytic functions *equivalent* if they are related in this way. We are ready for our classification result.

**Theorem 4.1.6.** *Let  $\underline{C} = (C_1, \dots, C_d)$  be a row contraction on a Hilbert space  $\mathcal{H}_C$ . Then there is a one-to-one correspondence between*

- (a) *unitary equivalence classes of subisometric liftings  $\underline{E}$  of  $\underline{C}$ ,*
- (b)  *$\underline{L} \otimes \mathbf{1}$ -invariant subspaces  $\mathcal{N}$  of  $\Gamma \otimes \mathcal{D}_C$ ,*

(c) multi-analytic inner operators  $M$  with symbols  $\Theta : \mathcal{D} \rightarrow \Gamma \otimes \mathcal{D}_C$  up to equivalence.

The correspondence is described above. In particular if  $\underline{E}$  is the lifting then  $\mathcal{D} = \mathcal{D}_E$ ,  $M = M_{C,E}$  with symbol  $\Theta = \Theta_{C,E}$  and (b)  $\leftrightarrow$  (c) is Beurling's theorem.

Theorem 4.1.6 shows that the characteristic function of a subisometric lifting characterizes the lifting up to unitary equivalence, justifying to call it characteristic.

*Proof.* (b)  $\leftrightarrow$  (c) is Beurling's theorem, see [Po89b]. We now show that the correspondence (a)  $\rightarrow$  (c) is well defined. Let  $\underline{E}$  on  $\mathcal{H}_E \supset \mathcal{H}_C$  and  $\underline{E}'$  on  $\mathcal{H}_{E'} \supset \mathcal{H}_C$  be two subisometric liftings of  $\underline{C}$  which are unitarily equivalent, i.e., there exists a unitary  $u : \mathcal{H}_E \rightarrow \mathcal{H}_{E'}$  such that  $u|_{\mathcal{H}_C} = \mathbf{1}|_{\mathcal{H}_C}$  and  $E'_i u = u E_i$  for  $i = 1, \dots, d$ . Clearly unitarily equivalent row contractions have unitarily equivalent mids and we can extend  $u$  (in a trivial way) to a unitary  $\hat{u}$  between the spaces  $\hat{\mathcal{H}}_E$  and  $\hat{\mathcal{H}}_{E'}$  of the mids  $\underline{V}^E$  and  $\underline{V}^{E'}$ , so we have

$$\hat{u} : \hat{\mathcal{H}}_E \rightarrow \hat{\mathcal{H}}_{E'} \text{ unitary, } \hat{u}|_{\mathcal{H}_E} = u, \quad V_i^{E'} \hat{u} = \hat{u} V_i^E \quad (i = 1, \dots, d)$$

Because  $\underline{E}, \underline{E}'$  are subisometric we also have unitaries  $W, W'$  such that

$$\begin{aligned} W : \hat{\mathcal{H}}_E &\rightarrow \hat{\mathcal{H}}_C, & V_i^C W &= W V_i^E, & W|_{\mathcal{H}_C} &= \mathbf{1}|_{\mathcal{H}_C} \\ W' : \hat{\mathcal{H}}_{E'} &\rightarrow \hat{\mathcal{H}}_C, & V_i^C W' &= W' V_i^{E'}, & W'|_{\mathcal{H}_C} &= \mathbf{1}|_{\mathcal{H}_C} \end{aligned}$$

If we now define

$$u_C := W' \hat{u} W^* : \hat{\mathcal{H}}_C \rightarrow \hat{\mathcal{H}}_C$$

then it follows that  $u_C$  commutes with the  $V_i^C$  for  $i = 1, \dots, d$ . To see

that, “chase” the following commuting diagram

$$\begin{array}{ccccc}
 & & \hat{\mathcal{H}}_C & \xrightarrow{u_C} & \hat{\mathcal{H}}_C \\
 & \nearrow W & \downarrow & & \nearrow W' \\
 \hat{\mathcal{H}}_E & \xrightarrow{\hat{u}} & \hat{\mathcal{H}}_{E'} & & \\
 \downarrow V_i^E & & \downarrow V_i^C & & \downarrow V_i^{E'} \\
 & \nearrow W & \hat{\mathcal{H}}_C & \xrightarrow{u_C} & \hat{\mathcal{H}}_C \\
 & & \downarrow & & \downarrow \\
 \hat{\mathcal{H}}_E & \xrightarrow{\hat{u}} & \hat{\mathcal{H}}_{E'} & & 
 \end{array} \tag{4.24}$$

Further, because  $W, W'$  and  $\hat{u}$  all fix  $\mathcal{H}_C$  pointwise the same is true for  $u_C$ , so we have also  $u_C|_{\mathcal{H}_C} = \mathbf{1}|_{\mathcal{H}_C}$ . But by minimality of  $\underline{V}^C$  we know that  $\hat{\mathcal{H}}_C$  is the closed linear span of vectors of the form  $V_\alpha^C h$  with  $\alpha \in \tilde{\Lambda}$ ,  $h \in \mathcal{H}_C$  and from

$$u_C V_\alpha^C h = V_\alpha^C u_C h = V_\alpha^C h$$

we infer that  $u_C = \mathbf{1}$ . Hence  $W = (u_C)^* W' \hat{u} = W' \hat{u}$ . Clearly  $\hat{u}$  maps  $e_0 \otimes \mathcal{D}_E \subset \hat{\mathcal{H}}_E$  onto  $e_0 \otimes \mathcal{D}_{E'} \subset \hat{\mathcal{H}}_{E'}$ , so if we define the unitary  $v := \hat{u}|_{\mathcal{D}_E} : \mathcal{D}_E \rightarrow \mathcal{D}_{E'}$  and use that  $\Theta = W|_{\mathcal{D}_E}$  and  $\Theta' = W'|_{\mathcal{D}_{E'}}$  we see that  $\Theta = \Theta' v$ , i.e., the characteristic functions are equivalent.

Conversely suppose that we are given a multi-analytic inner operator with symbol  $\Theta : \mathcal{D} \rightarrow \Gamma \otimes \mathcal{D}_C$ , as in (c). By (b)  $\leftrightarrow$  (c) (Beurling’s theorem) we have an invariant subspace  $\mathcal{N}$  which is associated to a subisometric lifting  $\underline{E}$  of  $\underline{C}$  and  $\mathcal{D} = \mathcal{D}_E$ , see the discussion preceding the theorem. It remains to show that if  $\Theta = \Theta' v$  with a unitary  $v : \mathcal{D}_E \rightarrow \mathcal{D}_{E'}$  for two subisometric liftings  $\underline{E}$  and  $\underline{E}'$  then  $\underline{E}$  and  $\underline{E}'$  are unitarily equivalent. Let  $W, W'$  be the corresponding unitaries from the subisometric lifting

property. Then

$$\begin{aligned}
 W'\mathcal{H}_{E'} &= \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C) \ominus W'(\Gamma \otimes \mathcal{D}_{E'}) \\
 &= \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C) \ominus M_{C,E'}(\Gamma \otimes v\mathcal{D}_E) \\
 &= \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C) \ominus M_{C,E}(\Gamma \otimes \mathcal{D}_E) = W\mathcal{H}_E,
 \end{aligned}$$

and we can define

$$U := (W')^* W|_{\mathcal{H}_E} : \mathcal{H}_E \rightarrow \mathcal{H}_{E'}.$$

Because for  $h \in \mathcal{H}_C$ ,  $Wh = h = W'h$  we have  $Uh = h$ . In general for  $h \in \mathcal{H}_E$  and  $i = 1, \dots, d$  (with  $p_E, p_{E'}$  orthogonal projections onto  $\mathcal{H}_E, \mathcal{H}_{E'}$ )

$$\begin{aligned}
 UE_i h &= (W')^* W E_i h = (W')^* W p_E V_i^E h = p_{E'} (W')^* W V_i^E h \\
 &= p_{E'} (W')^* V_i^C W h = p_{E'} V_i^{E'} (W')^* W h = E'_i U h,
 \end{aligned}$$

i.e.,  $\underline{E}$  and  $\underline{E}'$  are unitarily equivalent.  $\square$

There is an interesting variant of the classification if we not only give  $\underline{C}$  but also  $\underline{A}$ , i.e., if we consider liftings of  $\underline{C}$  by  $\underline{A}$ .

**Theorem 4.1.7.** *Let  $\underline{A}$  and  $\underline{C}$  be row contractions,  $\underline{A}$   $*$ -stable. There is a one-to-one correspondence between*

- (a) *unitary equivalence classes of subisometric liftings of  $\underline{C}$  by  $\underline{A}$*
- (b) *equivalence classes of isometries  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$ , two isometries considered equivalent if they have the same range*

*Proof.* The details of this correspondence have already been discussed in connection with Proposition 4.1.2. It is shown there how to construct an isometry  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  if a subisometric lifting of  $\underline{C}$  by  $\underline{A}$  is given, and conversely, how to use such an isometry to find a subisometric lifting.

The equivalence in (b) is chosen in such a way that two isometries are equivalent if and only if the associated invariant subspaces are the same, compare (4.12) and (4.14). Hence the result follows from Theorem 4.1.6.  $\square$

**Corollary 4.1.8.** *Let  $\underline{A}$  and  $\underline{C}$  be row contractions,  $\underline{A}$   $*$ -stable. A subisometric lifting of  $\underline{C}$  by  $\underline{A}$  exists if and only if*

$$\dim \mathcal{D}_{*,A} \leq \dim \mathcal{D}_C,$$

where  $\dim$  stands for the cardinality of an orthonormal basis. In the case  $\dim \mathcal{D}_{*,A} = \dim \mathcal{D}_C$  (minimal subisometric dilation in the terminology of [DF84]) the lifting is unique up to unitary equivalence.

## 4.2 Coisometric liftings

The theory of subisometric liftings turns out to be especially relevant in the case of coisometric row contractions and coisometric liftings. We start with definitions and elementary properties.

A row contraction  $\underline{C}$  on  $\mathcal{H}_1$  is called *coisometric* if  $\underline{C}\underline{C}^* = \mathbf{1}$ , i.e.,  $\sum_{i=1}^d C_i C_i^* = \mathbf{1}$ . It is easy to check that a lifting  $\underline{E}$  on  $\mathcal{H} = \mathcal{H}_C \oplus \mathcal{H}_A$  with block matrices

$$E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$$

(for all  $i = 1, \dots, d$ ) is coisometric if and only if  $\underline{C}$  is coisometric and

$$\underline{B}\underline{C}^* = 0, \quad \text{i.e.,} \quad \sum_{i=1}^d B_i C_i^* = 0, \quad (4.25)$$

$$\underline{A}\underline{A}^* + \underline{B}\underline{B}^* = \mathbf{1}, \quad \text{i.e.,} \quad \sum_{i=1}^d A_i A_i^* + \sum_{i=1}^d B_i B_i^* = \mathbf{1}. \quad (4.26)$$

**Theorem 4.2.1.** *Let  $\underline{A}$  and  $\underline{C}$  be row contractions,  $\underline{C}$  coisometric. Then there is a one-to-one correspondence between*

(a) *coisometric liftings  $\underline{E}$  of  $\underline{C}$  by  $\underline{A}$*

(b) *isometries  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$*

*Explicitly, if  $E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$  for  $i = 1, \dots, d$  provides a coisometric lifting  $\underline{E}$  of  $\underline{C}$  by  $\underline{A}$  then  $\gamma : D_{*,A}h \mapsto \underline{B}^*h \subset \mathcal{D}_C$  (for  $h \in \mathcal{H}_A$ ) is isometric.*

*Conversely, if  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  is isometric then with  $\underline{B}^* := \gamma D_{*,A}$  we obtain a coisometric lifting  $\underline{E}$  by  $E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$  for  $i = 1, \dots, d$ .*

*Proof.* Because  $\underline{C}$  is coisometric,  $D_C = \mathbf{1} - \underline{C}^*\underline{C}$  is the orthogonal projection onto the kernel of  $\underline{C}$ .

Let  $\underline{E}$  be a coisometric lifting of  $\underline{C}$  by  $\underline{A}$ . Then from (4.25) we have  $\underline{C}\underline{B}^* = (\underline{B}\underline{C}^*)^* = 0$  and hence  $\text{range}(\underline{B}^*) \subset \mathcal{D}_C$ .

Further for  $h \in \mathcal{H}_A$ , using (4.26)

$$\|D_{*,A}h\|^2 = \langle (\mathbf{1} - \underline{A}\underline{A}^*)h, h \rangle = \langle \underline{B}\underline{B}^*h, h \rangle = \|\underline{B}^*h\|^2$$

So there exist an isometry  $\gamma : \mathcal{D}_{*,A} \rightarrow \text{range}(\underline{B}^*) \subset \mathcal{D}_C$  with  $\gamma D_{*,A}h = \underline{B}^*h$  for all  $h \in \mathcal{H}_A$ .

Conversely, let  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  be an isometry and define  $\underline{B}^* := \gamma D_{*,A}$ . From  $\underline{C}|_{\mathcal{D}_C} = 0$  we obtain  $\underline{C}\underline{B}^* = 0$  or  $\underline{B}\underline{C}^* = 0$ , which is (4.25). Further

$$\underline{B}\underline{B}^* = D_{*,A}\gamma^*\gamma D_{*,A} = D_{*,A}^2 = \mathbf{1} - \underline{A}\underline{A}^*,$$

hence  $\underline{A}\underline{A}^* + \underline{B}\underline{B}^* = \mathbf{1}$ , which is (4.26). Hence with  $E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$ , for  $i = 1, \dots, d$ , we obtain a coisometric lifting  $\underline{E}$  of  $\underline{C}$  by  $\underline{A}$ .

Finally if  $\gamma, \gamma' : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  are two isometries and  $\gamma \neq \gamma'$  then  $\underline{B}^* \neq (\underline{B}')^*$  for  $\underline{B}^*, (\underline{B}')^*$  defined by  $\gamma, \gamma'$  as above. Hence the correspondence is one-to-one.  $\square$

**Corollary 4.2.2.** *Let  $\underline{A}$  and  $\underline{C}$  be row contractions,  $\underline{C}$  coisometric. A coisometric lifting  $\underline{E}$  of  $\underline{C}$  by  $\underline{A}$  exists if and only if*

$$\dim \mathcal{D}_{*,A} \leq \dim \mathcal{D}_C,$$

where  $\dim$  stands for the cardinality of an orthonormal basis..

Theorem 4.2.1 gives a kind of free parametrization of the coisometric liftings. Let us consider an elementary example.

$$\begin{aligned} \underline{c} = (c_1, \dots, c_d) \in \mathbb{C}^d, \quad \|\underline{c}\|^2 = \sum_{i=1}^d |c_i|^2 = 1 \quad (\text{unit sphere}) \quad (4.27) \\ \underline{a} = (a_1, \dots, a_d) \in \mathbb{C}^d, \quad \|\underline{a}\|^2 = \sum_{i=1}^d |a_i|^2 \leq 1 \quad (\text{unit ball}) \end{aligned}$$

Then we get a left lower corner  $\underline{b} = (b_1, \dots, b_d)$  for a coisometric lifting if  $\langle \underline{b}, \underline{c} \rangle = 0$  and  $\|\underline{a}\|^2 + \|\underline{b}\|^2 = 1$ , according to (4.25) and (4.26). Obviously the set of solutions for  $\underline{b}$  is the (complex) sphere with radius  $r = \sqrt{1 - \|\underline{a}\|^2}$  in the subspace orthogonal to  $\underline{c}$ . If  $\|\underline{a}\| = 1$  the solution is unique. We can check that the parametrization using isometries  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  as in Theorem 4.2.1 yields the same result.

Theorem 4.2.1 and Corollary 4.2.2 are even true if  $\underline{A}$  is not  $*$ -stable. If  $\underline{A}$  is  $*$ -stable then we should compare these results with those in Section 4.1. Note in particular that the formula  $\underline{B}^* = \gamma D_{*,A}$  in Theorem 4.2.1 and the formula  $\underline{B}^* = D_C^* \gamma D_{*,A}$  (1.10) are compatible because, as noted above, for  $\underline{C}$  coisometric the operator  $D_C^*$  is nothing but the embedding of  $\mathcal{D}_C$  into  $\oplus_{i=1}^d \mathcal{H}_C$  which is implicit in the formulation chosen in Theorem



4.2.1. Further comparison yields the following result which shows that subisometric liftings occur very naturally in the coisometric setting.

**Proposition 4.2.3.** *Let  $\underline{C}$  be a coisometric row contraction. A lifting of  $\underline{C}$  is a coisometric lifting by a  $*$ -stable  $\underline{A}$  if and only if it is subisometric.*

*Proof.* Using Theorem 4.2.1 we can replace the condition “coisometric” for the lifting by the existence of an isometry  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  such that  $\underline{B}^* = \gamma D_{*,A} = D_C^* \gamma D_{*,A}$ . Now Proposition 4.2.3 is a direct consequence of Proposition 4.1.2.  $\square$

In particular, for coisometric liftings by a  $*$ -stable  $\underline{A}$  there exists an associated invariant subspace and a characteristic function. In the special case  $\dim \mathcal{H}_C = 1$  this characteristic function was introduced in [DG07a] under the name “extended characteristic function”. For general  $\mathcal{H}_C$ , in view of Theorem 4.1.6, it is better to call it the characteristic function of the lifting (with  $\underline{C}$  given), as we have done in Definition 4.1.5.

## 4.3 Characteristic functions of reduced liftings

In this section we generalize the theory of characteristic functions for subisometric liftings from Section 4.1 and establish a setting that also includes the setting of Section 4.2.

Let  $\underline{C}$  be a row contraction on  $\mathcal{H}_C$  and  $\underline{E}$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  be a (contractive) lifting so that for all  $i = 1, \dots, d$

$$E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$$

Then as in (4.6) we have a mid  $\underline{V}^E$  on  $\mathcal{H}_E \oplus (\Gamma \otimes \mathcal{D}_E)$ . Clearly  $\underline{V}^E$  is an isometric lifting of  $\underline{C}$ , so the space of the mid  $\underline{V}^C$  can be embedded

as a subspace reducing the  $V_i^E$ . Let us encode this by introducing the restriction  $\underline{Y}$  on the orthogonal complement  $\mathcal{K}$  and a unitary  $W$  by

$$\begin{aligned} W : \mathcal{H}_E \oplus (\Gamma \otimes \mathcal{D}_E) &\rightarrow \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K} \\ \tilde{V}_i^E W &= W V_i^E, \quad W|_{\mathcal{H}_C} = \mathbf{1}|_{\mathcal{H}_C} \quad \text{with} \quad \tilde{\underline{V}}^E = \underline{V}^C \oplus \underline{Y} \end{aligned} \quad (4.28)$$

By omitting  $\mathcal{H}_C$  we also have a unitary (also denoted by  $W$ )

$$W : \mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E) \rightarrow (\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K} \quad (4.29)$$

and an isometric embedding  $\mathcal{H}_{A_*} := W\mathcal{H}_A \subset (\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K}$ . Further we obtain

$$\underline{B}^* = p_{\mathcal{H}_C}(\underline{V}^E)^*|_{\mathcal{H}_A} = p_{\mathcal{H}_C}[(\underline{V}^C)^* \oplus \underline{Y}^*]W|_{\mathcal{H}_A} = D_C^* p_{e_0 \otimes \mathcal{D}_C} W|_{\mathcal{H}_A} \quad (4.30)$$

where we used formula (4.6) for  $\underline{V}^C$ .

To proceed we need a few facts about the mid  $\underline{V}^A$  on  $\tilde{\mathcal{H}}_A$  of  $\underline{A}$ . We write its Wold decomposition as

$$\begin{aligned} \tilde{\mathcal{H}}_A &= (\Gamma \otimes \mathcal{D}_{*,A}) \oplus \mathcal{R}_A \\ V_i^A &= (L_i \otimes \mathbf{1}) \oplus R_i^A, \quad i = 1, \dots, d, \end{aligned} \quad (4.31)$$

where  $\mathcal{R}_A$  and  $\underline{R}^A$  stand for the residual part (cf. [Po89a]). The embedding of  $\mathcal{H}_A$  into  $\tilde{\mathcal{H}}_A$  can be written as

$$\mathcal{H}_A \ni h \mapsto \left( \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes D_{*,A} A_\alpha^* h \right) \oplus h_{\mathcal{R}} \quad (4.32)$$

Here  $h_{\mathcal{R}}$  belongs to the residual part  $\mathcal{R}_A$ . Compare [BDZ06] for a derivation of this decomposition via Stinespring's theorem. In fact, it is not difficult to check that a formula like (4.32) always reproduces the Wold decomposition above, compare also [FF90] for similar arguments for  $d = 1$ .

Note that the residual part vanishes if and only if  $\underline{A}$  is  $*$ -stable, so in this case we are back in the setting of Section 4.1.

Further we need the decomposition  $\mathcal{H}_A = \mathcal{H}_A^1 \oplus \mathcal{H}_A^0$  with  $\mathcal{H}_A^1$  the largest subspace invariant for the  $A_i^*$  and such that the restriction of  $\underline{A}^*$  is isometric, i.e.,

$$\mathcal{H}_A^1 := \{h \in \mathcal{H}_A : \sum_{|\alpha|=n} \|A_\alpha^* h\|^2 = \|h\|^2 \text{ for all } n \in \mathbb{N}\} \quad (4.33)$$

Then it is easy to check that  $\mathcal{H}_A^1 = \mathcal{H}_A \cap \mathcal{R}_A$  (cf. [Po89a], Proposition 2.9), but the position of  $\mathcal{H}_A^0$  may be complicated with respect to the decomposition (4.32) because  $\underline{A}$  restricted to  $\mathcal{H}_A^0$  may not be  $*$ -stable and in this case  $\mathcal{H}_A^0$  is not contained in  $\Gamma \otimes \mathcal{D}_{*,A}$ . In fact, if  $0 \neq h \in \mathcal{H}_A^0$  we only have

$$\sum_{|\alpha|=n} \|A_\alpha^* h\|^2 < \|h\|^2 \text{ for some } n \in \mathbb{N} \quad (4.34)$$

which (by definition) means that  $\underline{A}|_{\mathcal{H}_A^0}$  is *completely non-coisometric (c.n.c.)*, cf. [Po89a].

Now we look at  $\underline{A}$  and its mid  $\underline{V}^A$  embedded into the larger structure obtained from the lifting  $\underline{E}$ . Clearly  $\underline{V}^E$  restricted to  $\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)$  is an isometric dilation of  $\underline{A}$ , so  $\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)$  contains  $\tilde{\mathcal{H}}_A$  as a  $V_i^E$ -reducing subspace ( $i = 1, \dots, d$ ) which we still denote by  $\tilde{\mathcal{H}}_A$ . Using (4.29) we see that  $(\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K}$  contains the  $(L_i \otimes \mathbf{1}) \oplus Y_i$ -reducing subspace  $W\tilde{\mathcal{H}}_A$  and the restriction of  $(\underline{L} \otimes \mathbf{1}) \oplus \underline{Y}$  is a mid of  $\underline{A}$  (transferred to  $W\mathcal{H}_A$ ). Denoting the restriction of  $W$  to  $\tilde{\mathcal{H}}_A$  also by  $W$  we have (for  $i = 1, \dots, d$ )

$$W[(L_i \otimes \mathbf{1}) \oplus R_i^A] = WV_i^A = [(L_i \otimes \mathbf{1}) \oplus Y_i]W \quad (4.35)$$

Where is  $\mathcal{H}_{A*} = W\mathcal{H}_A$ ? Clearly

$$W\mathcal{H}_A^1 = W(\mathcal{H}_A \cap \mathcal{R}_A) \subset W\mathcal{R}_A \subset \mathcal{K}, \quad (4.36)$$

where the last inclusion follows from (4.35) and the fact that  $\underline{L} \otimes \mathbf{1}$  is  $*$ -stable. The position of  $W\mathcal{H}_A^0$  may be more complicated.

To organize the relevant data we use (4.31) together with the embedding of  $\tilde{\mathcal{H}}_A$  into  $\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)$  and (4.29) to define

$$\begin{aligned} M : \Gamma \otimes \mathcal{D}_{*,A} &\rightarrow \Gamma \otimes \mathcal{D}_C, \\ M &= P_{\Gamma \otimes \mathcal{D}_C} W|_{\Gamma \otimes \mathcal{D}_{*,A}} \end{aligned} \quad (4.37)$$

which is a multi-analytic operator. Then for  $h \in \mathcal{H}_A$

$$P_{e_0 \otimes \mathcal{D}_C} W h = P_{e_0 \otimes \mathcal{D}_C} M P_{\Gamma \otimes \mathcal{D}_{*,A}} h = P_{e_0 \otimes \mathcal{D}_C} M P_{e_0 \otimes \mathcal{D}_{*,A}} h$$

where for the first equality we used (4.36) and the second then follows from the fact that  $M$  is multi-analytic. But  $P_{e_0 \otimes \mathcal{D}_{*,A}} h = e_0 \otimes D_{*,A} h$  by (4.32) and we conclude that  $P_{e_0 \otimes \mathcal{D}_C} W|_{\mathcal{H}_A} : \mathcal{H}_A \rightarrow \mathcal{D}_C$  factors through  $\mathcal{D}_{*,A}$  in the sense that there exists a contraction  $\gamma := P_{e_0 \otimes \mathcal{D}_C} M|_{e_0 \otimes \mathcal{D}_{*,A}} : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  such that

$$P_{e_0 \otimes \mathcal{D}_C} W|_{\mathcal{H}_A} = \gamma D_{*,A} \quad (4.38)$$

In fact,  $\gamma$  is nothing but the 0-th Fourier coefficient of  $M$  in the sense of [Po03]. Combined with (4.30) we obtain

$$\underline{B}^* = D_C^* \gamma D_{*,A} : \mathcal{H}_A \rightarrow \bigoplus_{i=1}^d \mathcal{H}_C \quad (4.39)$$

This is one half of the following analogue for row contractions of Lemma 2.1 in Chap.IV of [FF90], which already has been discussed in the introduction, see in particular (4.3).

**Proposition 4.3.1.**  $\underline{E} = (E_1, \dots, E_d)$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  with block matrices

$$E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$$

(for  $i = 1, \dots, d$ ) is a row contraction if and only if  $\underline{C}$  and  $\underline{A}$  are row contractions and there exists a contraction  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  such that (4.39) holds.

*Proof.* Clearly if  $\underline{E}$  is a row contraction then  $\underline{C}$  and  $\underline{A}$  are row contractions. Above we have already given a (dilation) proof that if  $\underline{E}$  is contractive then  $\underline{B}$  satisfies (4.39) for a suitable contraction  $\gamma$ . To prove the converse, let  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  be a contraction and  $\underline{B}^*$  given as in (4.39). Then for  $x \in \mathcal{H}_C$ ,  $y \in \mathcal{H}_A$

$$\begin{aligned} |\langle x, \underline{C} \underline{B}^* y \rangle|^2 &= |\langle x, \underline{C} D_C^* \gamma D_{*,A} y \rangle|^2 = |\langle D_C \underline{C}^* x, \gamma D_{*,A} y \rangle|^2 \\ &\leq \|D_C \underline{C}^* x\|^2 \|\gamma D_{*,A} y\|^2 \leq \langle x, (\mathbf{1} - \underline{C} \underline{C}^*) x \rangle \langle y, (\mathbf{1} - \underline{A} \underline{A}^*) y \rangle \end{aligned}$$

which implies (see for example Exercise 3.2 in [Pau03]) that

$$0 \leq \begin{pmatrix} \mathbf{1} - \underline{C} \underline{C}^* & -\underline{C} \underline{B}^* \\ -\underline{B} \underline{C}^* & \mathbf{1} - \underline{A} \underline{A}^* \end{pmatrix} = \mathbf{1} - \underline{E} \underline{E}^*$$

hence  $\underline{E}$  is a row contraction.  $\square$

Let us go back to the lifting  $\underline{E}$  of  $\underline{C}$  by  $\underline{A}$ . The following definition is useful to analyze further the position of  $W\mathcal{H}_A$ .

**Definition 4.3.2.**  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  is called *resolving* if for all  $h \in \mathcal{H}_A$  we have

$$(\gamma D_{*,A} A_\alpha^* h = 0 \text{ for all } \alpha \in \tilde{\Lambda}) \Rightarrow (D_{*,A} A_\alpha^* h = 0 \text{ for all } \alpha \in \tilde{\Lambda})$$

Clearly if  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  is injective then it is resolving. Note that  $D_{*,A} A_\alpha^* h = 0$  for all  $\alpha \in \tilde{\Lambda}$  if and only if  $h \in \mathcal{H}_A^1$ , and so the intuitive meaning of ‘resolving’ is that ‘looking at  $\mathcal{H}_A$  through  $\gamma$ ’ still allows to detect whether  $h \in \mathcal{H}_A$  is in  $\mathcal{H}_A^1$  or not. More precisely,  $\gamma$  is resolving

if and only if for all  $h \in \mathcal{H}_A^0 = \mathcal{H}_A \ominus \mathcal{H}_A^1$  there exists  $\alpha \in \tilde{\Lambda}$  such that  $\gamma D_{*,A} A_\alpha^* h \neq 0$ . In particular if  $\underline{A}$  is c.n.c., i.e.  $\mathcal{H}_A^1 = \{0\}$ , then  $\gamma$  is resolving if and only if for all  $0 \neq h \in \mathcal{H}_A$  there exists  $\alpha \in \tilde{\Lambda}$  such that  $\gamma D_{*,A} A_\alpha^* h \neq 0$ .

**Lemma 4.3.3.** *The following assertions are equivalent*

- (a)  $\gamma$  is resolving.
- (b)  $W\mathcal{H}_A \cap \mathcal{K} \subset W\mathcal{H}_A^1$
- (c)  $W\mathcal{H}_A \cap \mathcal{K} = W\mathcal{H}_A^1$
- (d)  $(\Gamma \otimes \mathcal{D}_C) \vee W(\Gamma \otimes \mathcal{D}_E) = (\Gamma \otimes \mathcal{D}_C) \oplus (\mathcal{K} \ominus W\mathcal{H}_A^1)$

*Proof.* (b) says that for  $h \in \mathcal{H}_A \setminus \mathcal{H}_A^1$  the embedded  $Wh$  is not in  $\mathcal{K}$ , so not orthogonal to  $\Gamma \otimes \mathcal{D}_C$ , equivalently, there exists  $\alpha \in \tilde{\Lambda}$  such that

$$0 \neq P_{e_0 \otimes \mathcal{D}_C} [(L_\alpha^* \otimes \mathbf{1}) \oplus Y_\alpha^*] Wh = P_{e_0 \otimes \mathcal{D}_C} W(V_\alpha^A)^* h = \gamma D_{*,A} A_\alpha^* h$$

(where we used the embedding of the mid of  $\underline{A}$  and in particular (4.38)). By comparison with the comments following Definition 4.3.2 we conclude that (a) and (b) are equivalent. We noted in (4.36) that always  $W\mathcal{H}_A^1 \subset \mathcal{K}$ , so (b) and (c) are equivalent.

To get the equivalence of (c) and (d) note that  $x \in (\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K}$  is orthogonal to  $(\Gamma \otimes \mathcal{D}_C)$  and to  $W(\Gamma \otimes \mathcal{D}_E)$  if and only if  $x \in \mathcal{K}$  and  $x \in W\mathcal{H}_A$  (compare (4.29)). Hence the orthogonal complement of  $(\Gamma \otimes \mathcal{D}_C) \vee W(\Gamma \otimes \mathcal{D}_E)$  in  $(\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K}$  is in fact  $W\mathcal{H}_A \cap \mathcal{K}$ .  $\square$

**Definition 4.3.4.** *A lifting  $\underline{E}$  of  $\underline{C}$  by  $\underline{A}$  is called reduced if  $\underline{A}$  is c.n.c. (i.e.,  $\mathcal{H}_A^1 = \{0\}$ , see (4.34)) and  $\gamma$  is resolving.*

We have already seen two important classes of reduced liftings.

- 1) Subisometric liftings. Here  $\underline{A}$  is  $*$ -stable and  $\gamma$  is isometric, see Proposition 4.1.2.
- 2) Coisometric liftings by  $\underline{A}$  c.n.c. Here  $\gamma$  is isometric by Theorem 4.2.1.

Note that by Proposition 4.2.3 the coisometric liftings by  $*$ -stable  $\underline{A}$  are exactly the intersection of cases 1) and 2).

**Lemma 4.3.5.** *The following assertions are equivalent*

- (a)  $\underline{E}$  is reduced.
- (b)  $\{h \in \mathcal{H}_A : \gamma D_{*,A} A_\alpha^* h = 0 \text{ for all } \alpha \in \tilde{\Lambda}\} = \{0\}$
- (c)  $W\mathcal{H}_A \cap \mathcal{K} = \{0\}$

*Proof.* If  $\gamma$  is resolving then (by definition) the space given in (b) is contained in  $\mathcal{H}_A^1$ . Hence (a) implies (b). Also, from (b) we first conclude that  $\mathcal{H}_A^1 = \{h \in \mathcal{H}_A : D_{*,A} A_\alpha^* h = 0 \text{ for all } \alpha \in \tilde{\Lambda}\} = \{0\}$  and then that  $\gamma$  is resolving, so (b) implies (a). If we have (c) then by Lemma 4.3.3(b)  $\gamma$  is resolving and then by Lemma 4.3.3(c)  $\underline{A}$  is c.n.c., so we have (a). Given (a), Lemma 4.3.3(c) implies (c).  $\square$

If  $\gamma D_{*,A} A_\alpha^* h = 0$  for all  $\alpha \in \tilde{\Lambda}$  then by (4.39) we conclude that  $A_\alpha^* h \in \ker \underline{B}^* = (\text{range } \underline{B})^\perp$ . Hence vectors in the space  $\{h \in \mathcal{H}_A : \gamma D_{*,A} A_\alpha^* h = 0 \text{ for all } \alpha \in \tilde{\Lambda}\}$  do not contribute in any way to the interaction between  $\mathcal{H}_A$  and  $\mathcal{H}_C$  via  $\underline{B}^*$ , and it is no great loss to concentrate on liftings where this space has been removed. By Lemma 4.3.5(b), in doing this we obtain exactly the reduced liftings. This also explains our terminology.

For reduced liftings we can successfully develop a theory of characteristic functions.

**Definition 4.3.6.** Let  $\underline{E}$  be a reduced lifting of  $\underline{C}$  by  $\underline{A}$ . We call the multi-analytic operator

$$\begin{aligned} M_{C,E} : \Gamma \otimes \mathcal{D}_E &\rightarrow \Gamma \otimes \mathcal{D}_C, \\ M_{C,E} &= P_{\Gamma \otimes \mathcal{D}_C} W|_{\Gamma \otimes \mathcal{D}_E} \end{aligned} \quad (4.40)$$

(or its symbol  $\Theta_{C,E} : \mathcal{D}_E \rightarrow \Gamma \otimes \mathcal{D}_C$ ) the characteristic function of the lifting  $\underline{E}$ .

Using the characteristic function we can develop a theory of functional models for reduced liftings. The idea is similar as in the case of characteristic functions for c.n.c. row contractions, see [Po89b].

Let  $\underline{E}$  be a reduced lifting of  $\underline{C}$  by  $\underline{A}$ . From  $\underline{A}$  c.n.c. we obtain  $\mathcal{H}_A^1 = \{0\}$  and then Lemma 4.3.3 gives

$$(\Gamma \otimes \mathcal{D}_C) \bigvee W(\Gamma \otimes \mathcal{D}_E) = (\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K} \quad (4.41)$$

With the definition

$$\Delta_{C,E} := (\mathbf{1} - M_{C,E}^* M_{C,E})^{\frac{1}{2}} : \Gamma \otimes \mathcal{D}_E \rightarrow \Gamma \otimes \mathcal{D}_E \quad (4.42)$$

we obtain for  $x \in \Gamma \otimes \mathcal{D}_E$

$$\begin{aligned} \|P_{\mathcal{K}} Wx\|^2 &= \|(\mathbf{1} - P_{\Gamma \otimes \mathcal{D}_C}) Wx\|^2 = \|x\|^2 - \|P_{\Gamma \otimes \mathcal{D}_C} Wx\|^2 \\ &= \|x\|^2 - \|M_{C,E} x\|^2 = \|\Delta_{C,E} x\|^2 \end{aligned} \quad (4.43)$$

This means that we can isometrically identify  $\mathcal{K}$  with  $\overline{\Delta_{C,E}(\Gamma \otimes \mathcal{D}_E)}$  and with this identification we have

$$W\mathcal{H}_A = [(\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K}] \ominus W(\Gamma \otimes \mathcal{D}_E) \quad (4.44)$$

$$= [(\Gamma \otimes \mathcal{D}_C) \oplus \overline{\Delta_{C,E}(\Gamma \otimes \mathcal{D}_E)}] \ominus \{M_{C,E} x \oplus \Delta_{C,E} x : x \in \Gamma \otimes \mathcal{D}_C\}$$

which is a kind of functional model.



**Theorem 4.3.7.** *Let  $\underline{C}$  be a row contraction. Reduced liftings  $\underline{E}$  and  $\underline{E}'$  of  $\underline{C}$  are unitarily equivalent if and only if their characteristic functions  $M_{C,E}$  and  $M_{C,E'}$  are equivalent.*

Recall that  $M_{C,E}$  and  $M_{C,E'}$  are equivalent if there exists a unitary  $v : \mathcal{D}_E \rightarrow \mathcal{D}_{E'}$  such that their symbols satisfy  $\Theta_{C,E} = \Theta_{C,E'} v$ . Compared with the analogous result for subisometric liftings contained in Theorem 4.1.6 the modifications necessary to prove Theorem 4.3.7 are technical and straightforward, so we omit the proof. The important thing to recognize is that, if a lifting  $\underline{E}$  is reduced, we have the functional model (4.44) for it which is built only from  $\underline{C}$  and from the characteristic function  $M_{C,E}$ .

Conversely, if  $\underline{C}$  on  $\mathcal{H}_C$  is a row contraction and

$$\tilde{M} : \Gamma \otimes \mathcal{D} \rightarrow \Gamma \otimes \mathcal{D}_C$$

is an arbitrary contractive multi-analytic function (where  $\mathcal{D}$  is any Hilbert space), then we can define

$$\Delta : (\mathbf{1} - \tilde{M}^* \tilde{M})^{\frac{1}{2}} : \Gamma \otimes \mathcal{D} \rightarrow \Gamma \otimes \mathcal{D}$$

$$\tilde{\mathcal{H}} := \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C) \oplus \overline{\Delta(\Gamma \otimes \mathcal{D})}$$

$$\tilde{W} : \Gamma \otimes \mathcal{D} \rightarrow (\Gamma \otimes \mathcal{D}_C) \oplus \overline{\Delta(\Gamma \otimes \mathcal{D})}, \quad x \mapsto \tilde{M}x \oplus \Delta x$$

$\tilde{W}$  is isometric and by introducing a copy  $\mathcal{H}_A$  of the orthogonal complement of  $\tilde{W}(\Gamma \otimes \mathcal{D})$  we can extend  $\tilde{W}$  to a unitary

$$\tilde{W} : \mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}) \rightarrow (\Gamma \otimes \mathcal{D}_C) \oplus \overline{\Delta(\Gamma \otimes \mathcal{D})}$$

Let  $\tilde{\underline{V}} = (\tilde{V}_1, \dots, \tilde{V}_d)$  be defined on  $\tilde{\mathcal{H}}$  by  $\tilde{V}_i := V_i^C \oplus Y_i$  (for  $i = 1, \dots, d$ ), where  $\underline{V}^C$  is the mid of  $\underline{C}$  on  $\mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C)$  (4.6) and  $Y_i$  is given by

$$Y_i \Delta x := \Delta(L_i \otimes \mathbf{1})x \quad (\text{where } x \in \Gamma \otimes \mathcal{D})$$

It is not difficult to check that  $\underline{Y}$  (and hence also  $\underline{\tilde{V}}$ ) is a row contraction consisting of isometries with orthogonal ranges (i.e., a row isometry). Further

$$\tilde{W}(\Gamma \otimes \mathcal{D}) = \{\tilde{M}x \oplus \Delta x, x \in \Gamma \otimes \mathcal{D}\}$$

is invariant for the  $\tilde{V}_i$ . With  $E_i^* := \tilde{V}_i^*|_{\mathcal{H}_C \oplus \tilde{W}\mathcal{H}_A}$ ,  $A_i^* := \tilde{V}_i^*|_{\tilde{W}\mathcal{H}_A}$  for  $i = 1, \dots, d$  we obtain a contractive lifting  $\underline{E}$  of  $\underline{C}$  by  $\underline{A}$  which we may call *the lifting associated to the multi-analytic function  $\tilde{M}$* . The following result gives another justification for considering reduced liftings.

**Proposition 4.3.8.** *The contractive lifting  $\underline{E}$  associated to a row contraction  $\underline{C}$  and a contractive multi-analytic function  $\tilde{M} : \Gamma \otimes \mathcal{D} \rightarrow \Gamma \otimes \mathcal{D}_C$  (where  $\mathcal{D}$  is any Hilbert space) is reduced.*

*Proof.* By Lemma 4.3.5 it is enough to show that any vector  $y \in \tilde{W}\mathcal{H}_A$  which is orthogonal to  $\Gamma \otimes \mathcal{D}_C$  is the zero vector. But  $y \in \tilde{W}\mathcal{H}_A$  means that  $y$  is orthogonal to  $\tilde{M}x \oplus \Delta x$  for all  $x \in \Gamma \otimes \mathcal{D}$  and  $y$  orthogonal to  $\Gamma \otimes \mathcal{D}_C$  means that  $y \in 0 \oplus \overline{\Delta(\Gamma \otimes \mathcal{D})}$ . Hence indeed  $y = 0$ .  $\square$

Proposition 4.3.8 shows that the theory of characteristic functions cannot be extended beyond reduced liftings. Note that  $\tilde{M}$  is not necessarily the characteristic function of the associated lifting  $\underline{E}$  and we used  $\sim$  to indicate this. It is an interesting question which intrinsic properties of  $\tilde{M}$  guarantee that it is the characteristic function. We leave this as an open problem.

## 4.4 Properties of the characteristic function

First we shall compute an explicit expression for the characteristic function of a reduced lifting. We continue to use the notation of the previous section

and consider a reduced lifting  $\underline{E}$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  of  $\underline{C}$  on  $\mathcal{H}_C$  by  $\underline{A}$  on  $\mathcal{H}_A$ . As in (4.35) the row isometry  $(\underline{L} \otimes \mathbf{1}) \oplus \underline{Y}$  on  $(\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K}$  restricts to a mid of  $\underline{A}$  (transferred to  $W\mathcal{H}_A$ ). So we have for all  $\alpha \in \tilde{\Lambda}$  and  $h \in \mathcal{H}_A$

$$[(L_\alpha^* \otimes \mathbf{1}) \oplus Y_\alpha^*]Wh = W A_\alpha^* h \quad (4.45)$$

Using (4.38) we infer that

$$\gamma D_{*,A} A_\alpha^* h = P_{e_0 \otimes \mathcal{D}_C} W A_\alpha^* h = P_{e_0 \otimes \mathcal{D}_C} [(L_\alpha^* \otimes \mathbf{1}) \oplus Y_\alpha^*] Wh = P_{e_\alpha \otimes \mathcal{D}_C} Wh \quad (4.46)$$

which yields a Poisson kernel type formula, compare (4.11):

$$P_{\Gamma \otimes \mathcal{D}_C} Wh = \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes \gamma D_{*,A} A_\alpha^* h \quad (4.47)$$

To compute the symbol  $\Theta_{C,E}$  of the characteristic function we define  $d_h^i := (V_i^E - E_i)h = e_0 \otimes (D_E)_i h$  and use the identification of  $\mathcal{D}_E$  with the closed linear span of all  $d_h^i$  with  $i = 1, \dots, d$  and  $h \in \mathcal{H}_E$ , see (4.6). Then, using (4.28) and the Definition 4.3.6 of  $\Theta_{C,E}$ , we obtain

$$\Theta_{C,E} d_h^i = P_{\Gamma \otimes \mathcal{D}_C} W(V_i^E - E_i)h = P_{\Gamma \otimes \mathcal{D}_C} V_i^C P_{\mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C)} Wh - P_{\Gamma \otimes \mathcal{D}_C} W E_i h \quad (4.48)$$

We distinguish two cases.

**Case I:**  $h \in \mathcal{H}_C$ .

$$P_{\Gamma \otimes \mathcal{D}_C} V_i^C P_{\mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C)} Wh = P_{\Gamma \otimes \mathcal{D}_C} V_i^C h = [e_0 \otimes (D_C)_i h] \quad \text{by (4.6)}$$

$$P_{\Gamma \otimes \mathcal{D}_C} W E_i h = P_{\Gamma \otimes \mathcal{D}_C} W(C_i h \oplus B_i h) = \sum_{\alpha} e_\alpha \otimes \gamma D_{*,A} A_\alpha^* B_i h \quad \text{by (4.47)}$$

and thus

$$\Theta_{C,E} d_h^i = e_0 \otimes [(D_C)_i h - \gamma D_{*,A} B_i h] - \sum_{|\alpha| \geq 1} e_\alpha \otimes \gamma D_{*,A} A_\alpha^* B_i h \quad (4.49)$$

**Case II:**  $h \in \mathcal{H}_A$ .

$$\begin{aligned} P_{\Gamma \otimes \mathcal{D}_C} V_i^C P_{\mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C)} W h &= V_i^C P_{\Gamma \otimes \mathcal{D}_C} W h \\ &= (L_i \otimes \mathbf{1}) P_{\Gamma \otimes \mathcal{D}_C} W h = \sum_{\alpha} e_i \otimes e_{\alpha} \otimes \gamma D_{*,A} A_{\alpha}^* h \end{aligned}$$

$$P_{\Gamma \otimes \mathcal{D}_C} W E_i h = P_{\Gamma \otimes \mathcal{D}_C} W A_i h = \sum_{\beta} e_{\beta} \otimes \gamma D_{*,A} A_{\beta}^* A_i h$$

Note that for  $h \in \mathcal{H}_A$  we have  $(D_A)_i h = (D_E)_i h$  (which we identify with  $d_h^i$ ) because  $\underline{E}$  is an extension of  $\underline{A}$ . With  $P_j$  the orthogonal projection onto the  $j$ -th component we obtain

$$\begin{aligned} \Theta_{C,E} d_h^i &= -e_0 \otimes \gamma D_{*,A} A_i h + \sum_{j=1}^d e_j \otimes \sum_{\alpha} e_{\alpha} \otimes \gamma D_{*,A} A_{\alpha}^* (\delta_{ji} \mathbf{1} - A_j^* A_i) h \\ &= -e_0 \otimes \gamma \underline{A} (D_A)_i h + \sum_{j=1}^d e_j \otimes \sum_{\alpha} e_{\alpha} \otimes \gamma D_{*,A} A_{\alpha}^* P_j D_A (D_A)_i h \\ &= -e_0 \otimes \gamma \sum_{j=1}^d A_j P_j d_h^i + \sum_{j=1}^d e_j \otimes \sum_{\alpha} e_{\alpha} \otimes \gamma D_{*,A} A_{\alpha}^* P_j D_A d_h^i \end{aligned} \tag{4.50}$$

We note that if  $\gamma$  is omitted from (4.50) then we obtain exactly Popescu's definition of the characteristic function of the (c.n.c.) row contraction  $\underline{A}$  as given in [Po89b]. Hence Case II is essentially the characteristic function of  $\underline{A}$ , contractively embedded by  $\gamma$ . In a special case this has been observed in [DG07a] and, because this special case was subisometric and hence  $\gamma$  isometric,  $\Theta$  was called an extended characteristic function. (4.50) generalizes this idea.

Let us now illustrate how the characteristic function factorizes for iterated liftings. Assume that  $\tilde{\underline{E}}$  on  $\mathcal{H}_{\tilde{E}}$  is a two step lifting of the row contraction  $\underline{C}$  on  $\mathcal{H}_C$ , i.e.,  $\underline{E}$  on  $\mathcal{H}_E$  with  $E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$  (for  $i = 1, \dots, d$ ) is a contractive lifting of  $\underline{C}$  on  $\mathcal{H}_C$  by  $\underline{A}$  on  $\mathcal{H}_A$  (as before) and  $\tilde{\underline{E}}$  on  $\mathcal{H}_{\tilde{E}}$  with  $\tilde{E}_i = \begin{pmatrix} E_i & 0 \\ * & \tilde{A}_i \end{pmatrix}$  (for  $i = 1, \dots, d$ ) is a contractive lifting of  $\underline{E}$  on  $\mathcal{H}_E$  by  $\tilde{\underline{A}}$  on  $\mathcal{H}_{\tilde{A}}$ . Then  $\mathcal{H}_{\tilde{E}} = \mathcal{H}_E \oplus \mathcal{H}_{\tilde{A}} = \mathcal{H}_C \oplus \mathcal{H}_A \oplus \mathcal{H}_{\tilde{A}}$  and with respect to this decomposition

$$\tilde{E}_i = \begin{pmatrix} C_i & 0 & 0 \\ * & A_i & 0 \\ * & * & \tilde{A}_i \end{pmatrix} \quad (4.51)$$

‘\*’ stands for entries which we do not need to name explicitly.

**Theorem 4.4.1.** *If the liftings  $\underline{E}$  of  $\underline{C}$  and  $\tilde{\underline{E}}$  of  $\underline{E}$  are reduced then also the lifting  $\tilde{\underline{E}}$  of  $\underline{C}$  is reduced, and the characteristic functions factorize:*

$$M_{C, \tilde{E}} = M_{C, E} M_{E, \tilde{E}}. \quad (4.52)$$

*Proof.* As in (4.28) we obtain the following unitaries from the given liftings:

$$W : \mathcal{H}_E \oplus (\Gamma \otimes \mathcal{D}_E) \rightarrow \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K}$$

$$\tilde{W} : \mathcal{H}_{\tilde{E}} \oplus (\Gamma \otimes \mathcal{D}_{\tilde{E}}) \rightarrow \mathcal{H}_E \oplus (\Gamma \otimes \mathcal{D}_E) \oplus \tilde{\mathcal{K}}$$

satisfying

$$W V_i^E = (V_i^C \oplus Y_i) W$$

$$\tilde{W} V_i^{\tilde{E}} = (V_i^E \oplus \tilde{Y}_i) \tilde{W}$$

We can define another unitary

$$Z := (W \otimes \mathbf{1}_{\tilde{\mathcal{K}}}) \tilde{W} : \mathcal{H}_{\tilde{E}} \oplus (\Gamma \otimes \mathcal{D}_{\tilde{E}}) \rightarrow \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K} \oplus \tilde{\mathcal{K}} \quad (4.53)$$

satisfying

$$Z V_i^{\tilde{E}} = (V_i^C \oplus Y_i \oplus \tilde{Y}_i) Z \quad (4.54)$$

Note further that  $W, \tilde{W}$  and hence also  $Z$  act identically on  $\mathcal{H}_C$ . By assumption the liftings  $\underline{E}$  of  $\underline{C}$  and  $\tilde{\underline{E}}$  of  $\underline{E}$  are reduced and we have characteristic functions

$$M_{C,E} = P_{\Gamma \otimes \mathcal{D}_C} W|_{\Gamma \otimes \mathcal{D}_E}$$

$$M_{E,\tilde{E}} = P_{\Gamma \otimes \mathcal{D}_E} \tilde{W}|_{\Gamma \otimes \mathcal{D}_{\tilde{E}}}$$

They can be composed to yield a multi-analytic operator

$$M := M_{C,E} M_{E,\tilde{E}} : \Gamma \otimes \mathcal{D}_{\tilde{E}} \rightarrow \Gamma \otimes \mathcal{D}_C$$

Using (4.53) it is easily checked that

$$M = P_{\Gamma \otimes \mathcal{D}_C} Z|_{\Gamma \otimes \mathcal{D}_{\tilde{E}}}$$

We conclude by (4.54) that the lifting  $\tilde{\underline{E}}$  of  $\underline{C}$  is associated to  $M$  and hence, by Proposition 4.3.8, this lifting is reduced. In fact, comparing with Definition 4.3.6, we see that  $M$  is the characteristic function, i.e.,  $M = M_{C,\tilde{E}}$ .  $\square$

## 4.5 Applications to completely positive maps

If  $\underline{T} = (T_1, \dots, T_d)$  is a row contraction on a Hilbert space  $\mathcal{K}$  then we denote by  $\Phi_T$  the corresponding (normal) completely positive map on

$\mathcal{B}(\mathcal{K})$  given by

$$\Phi_T(\cdot) = \sum_{i=1}^d T_i \cdot T_i^* \quad (4.55)$$

If  $d = \infty$  this should be understood as a SOT-limit. See for example [Pau03] for the general theory of completely positive maps, we shall only work with the concrete representation (4.55). The fact that  $\underline{T}$  is a row contraction implies that  $\Phi_T(\mathbf{1}) \leq \mathbf{1}$ , i.e.,  $\Phi_T$  is contractive. It is unital ( $\Phi_T(\mathbf{1}) = \mathbf{1}$ ) if and only if  $\underline{T}$  is coisometric.

If  $\underline{E}$  is a contractive lifting of  $\underline{C}$  by  $\underline{A}$ , i.e.,  $E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$  (for  $i = 1, \dots, d$ ) then an elementary computation shows that

$$\Phi_E \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \sum_{i=1}^d \begin{pmatrix} C_i X_{11} C_i^* & C_i X_{11} B_i^* + C_i X_{12} A_i^* \\ B_i X_{11} C_i^* & B_i X_{11} B_i^* + B_i X_{12} A_i^* \\ + A_i X_{21} C_i^* & + A_i X_{21} B_i^* + A_i X_{22} A_i^* \end{pmatrix} \quad (4.56)$$

with  $X_{11} \in \mathcal{B}(\mathcal{H}_C)$ ,  $X_{12} \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}_C)$ ,  $X_{21} \in \mathcal{B}(\mathcal{H}_C, \mathcal{H}_A)$ ,  $X_{22} \in \mathcal{B}(\mathcal{H}_A)$ . We denote by  $p_C = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}$  and  $p_A = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$  the orthogonal projections onto  $\mathcal{H}_C$  and  $\mathcal{H}_A$ . The following facts are immediate from (4.56).

$$p_C (\Phi_E)^n \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} |_{\mathcal{H}_C} = (\Phi_C)^n(X) \quad (4.57)$$

(for  $n \in \mathbb{N}_0$  and  $X \in \mathcal{B}(\mathcal{H}_C)$ )

$$\Phi_E \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_A(Y) \end{pmatrix} \quad (4.58)$$

(for  $Y \in \mathcal{B}(\mathcal{H}_A)$ ). So  $\Phi_E$  is a kind of (power) dilation of  $\Phi_C$  (4.57) and an extension of  $\Phi_A$  (4.58).

**Definition 4.5.1.** *If  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$ ,  $\Phi_E : \mathcal{B}(\mathcal{H}_E) \rightarrow \mathcal{B}(\mathcal{H}_E)$ ,  $\Phi_C : \mathcal{B}(\mathcal{H}_C) \rightarrow \mathcal{B}(\mathcal{H}_C)$ ,  $\Phi_A : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_A)$  are contractive normal completely positive maps such that (4.57) and (4.58) are valid then we say that  $\Phi_E$  is a contractive lifting of  $\Phi_C$  by  $\Phi_A$ .*

We have seen that a contractive lifting of row contractions gives rise to a contractive lifting of completely positive maps. The converse is also true: Let us assume (4.58). If  $\Phi_E(\cdot) = \sum_{i=1}^d E_i \cdot E_i^*$  and we write  $E_i = \begin{pmatrix} C_i & D_i \\ B_i & A_i \end{pmatrix}$  for the moment, then

$$\Phi_E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^d D_i D_i^* & * \\ * & * \end{pmatrix}$$

and (4.58) implies that all the  $D_i$  are zero, i.e., we have a lifting of row contractions. So actually (4.58) implies (4.57) with some  $\Phi_C$ .

Note that if  $\underline{E} = \underline{V}^C$ , the mid of  $\underline{C}$ , then  $\Phi_E$  is a  $*$ -homomorphism and (4.57) shows that the powers of  $\Phi_E$  are a homomorphic dilation of the completely positive semigroup formed by powers of  $\Phi_C$ . See [BP94, Ar03, Go04] for further information about this kind of dilation theory.

The discussion above shows that we can use our theory of liftings for row contractions to study liftings of completely positive maps. If  $\underline{E}$  is a reduced lifting of  $\underline{C}$  by  $\underline{A}$  then we have a characteristic function  $M_{C,E}$ . It is well known (see for example [Pau03, Go04]) that in the decomposition  $\Phi_E = \sum_{i=1}^d E_i \cdot E_i^*$  the tuple  $(E_1, \dots, E_d)$  is not uniquely determined and that  $\sum_{i=1}^d E'_i \cdot (E'_i)^*$  describes the same map if and only if  $\underline{E}'$  is obtained from  $\underline{E}$  by multiplication with a unitary  $d \times d$ -matrix (with complex entries). This does not change the characteristic function because the latter



is defined as an intertwiner between objects which are transformed in the same way. Hence it is possible to think of  $M_{C,E}$  also as the characteristic function of a reduced lifting  $\Phi_E$  of  $\Phi_C$  by  $\Phi_A$ . (Of course we call this lifting reduced if the corresponding lifting of row contractions is reduced.) Theorem 4.3.7 translates immediately into

**Corollary 4.5.2.** *Given  $\Phi_C$ , two reduced liftings  $\Phi_E$  resp.  $\Phi_{E'}$  of  $\Phi_C$  by  $\Phi_A$  resp.  $\Phi_{A'}$  are conjugate, i.e.*

$$\Phi_E = U^* \Phi_{E'} (UXU^*) U$$

with a unitary  $U : \mathcal{H}_E \rightarrow \mathcal{H}_{E'}$  such that  $U|_{\mathcal{H}_C} = \mathbf{1}|_{\mathcal{H}_C}$ , if and only if the corresponding characteristic functions are equivalent.

Corollary 4.5.2 generalizes Corollary 6.3 in [DG07a] where  $\dim \mathcal{H}_C = 1$ .

In the following we confine ourselves mainly to liftings which are coisometric and subisometric and give some concrete and useful results about the corresponding completely positive maps.

**Lemma 4.5.3.** *Let  $\underline{E}$  be a contractive lifting of a row contraction  $\underline{C}$  by a \*-stable row contraction  $\underline{A}$ . Then for all  $X_{12}, X_{21}, X_{22}$*

$$\Phi_E^n \begin{pmatrix} 0 & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \rightarrow 0$$

as  $n \rightarrow \infty$  (SOT).

*Proof.*  $\Phi_E^n(p_A)$  decreases to zero in the strong operator topology because of (4.58) and the assumption that  $\underline{A}$  is \*-stable. Then also  $\Phi_E^n \begin{pmatrix} 0 & 0 \\ 0 & X_{22} \end{pmatrix} \rightarrow 0$ , first for  $0 \leq X_{22} \leq \|X_{22}\| p_A$ , then for general  $X_{22}$  by writing it as a

linear combination of positive elements. Using the Kadison-Schwarz inequality for completely positive maps (cf. [Ch74] or [Pau03], Chapter 3) we obtain

$$\Phi_E^n \begin{pmatrix} 0 & 0 \\ X_{12}^* & 0 \end{pmatrix} \Phi_E^n \begin{pmatrix} 0 & X_{12} \\ 0 & 0 \end{pmatrix} \leq \Phi_E^n \begin{pmatrix} 0 & 0 \\ 0 & X_{12}^* X_{12} \end{pmatrix} \rightarrow 0$$

and hence  $\Phi_E^n \begin{pmatrix} 0 & X_{12} \\ 0 & 0 \end{pmatrix} \rightarrow 0$ . Similarly  $\Phi_E^n \begin{pmatrix} 0 & 0 \\ X_{21} & 0 \end{pmatrix} \rightarrow 0$ .  $\square$

**Theorem 4.5.4.** *Suppose the row coisometry  $\underline{E}$  is a lifting of  $\underline{C}$  by  $\underline{A}$ . Then the following assertions are equivalent:*

- (a) *The lifting is subisometric.*
- (b)  *$\underline{A}$  is  $*$ -stable.*
- (c)  $(\Phi_E)^n(p_C) \rightarrow \mathbf{1} \quad (n \rightarrow \infty, SOT)$
- (d) *There is an order isomorphism between the fixed point sets of  $\Phi_E$  and of  $\Phi_C$  given by*

$$\kappa : X \mapsto p_C X p_C \tag{4.59}$$

*In this case,  $\kappa$  is isometric on selfadjoint elements. If  $x$  is a fixed point of  $\Phi_C$  then we can reconstruct the preimage  $\kappa^{-1}(x)$  as the SOT-limit*

$$\lim_{n \rightarrow \infty} (\Phi_E)^n \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \tag{4.60}$$

Recall further that by the results of Section 4.2 the liftings in Theorem 4.5.4 are parametrized by  $*$ -stable row contractions  $\underline{A}$  with  $\dim \mathcal{D}_{*,A} \leq \dim \mathcal{D}_C$  together with isometries  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  and that they can be explicitly constructed from these data. Theorem 4.5.4(d) tells us that

(exactly) for such liftings the maps  $\Phi_E$  and  $\Phi_C$  have closely related properties in terms of their fixed points. We can identify this useful situation by checking the convenient conditions (b) or (c).

*Proof.* By Proposition 4.2.3 a coisometric lifting  $\underline{E}$  of  $\underline{C}$  by  $\underline{A}$  is subisometric if and only if  $\underline{A}$  is  $*$ -stable. Using (4.58) the latter means that

$$(\Phi_E)^n(p_A) \rightarrow 0 \quad (n \rightarrow \infty, \text{ SOT}),$$

which is equivalent to (c) because  $\Phi_E$  is unital. Hence  $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ .

If  $X = \begin{pmatrix} x & * \\ * & * \end{pmatrix}$  is a fixed point of  $\Phi_E$  then it is immediate from (4.56) that  $x$  is a fixed point of  $\Phi_C$ . Hence  $\kappa : X \mapsto p_C X p_C$  indeed maps fixed points of  $\Phi_E$  to fixed points of  $\Phi_C$ . (This is true for all contractive liftings.) Now assume (a), i.e., the lifting is subisometric. Then

$$X = \Phi_E(X) = \lim_{n \rightarrow \infty} (\Phi_E)^n(X) = \lim_{n \rightarrow \infty} (\Phi_E)^n \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix},$$

where the last equality follows from Lemma 4.5.3. Hence  $\kappa$  is injective.

Let  $\underline{V} = (V_1, \dots, V_d)$  simultaneously serve as mid for  $\underline{C}$  and  $\underline{E}$ . Then Theorem 4.5.1 in [BJKW00] or Lemma 4.6.4 in the Appendix of this paper show that for every fixed point  $x$  of  $\Phi_C$  there exists  $A'$  in the commutant of  $V_1, \dots, V_d$  such that  $p_C A' p_C = x$ . Define  $X := p_E A' p_E$ , where  $p_E$  is the orthogonal projection onto  $\mathcal{H}_E$ . Then, using the lifting property  $E_i p_E = p_E V_i$  for  $i = 1, \dots, d$  for the mid and the fact that  $\sum_{i=1}^d V_i V_i^* = \mathbf{1}$  (because  $\underline{E}$  is coisometric also  $\underline{V}$  is coisometric), we find that

$$\begin{aligned} \Phi_E(X) &= \sum_{i=1}^d E_i X E_i^* = \sum_{i=1}^d E_i p_E A' p_E E_i^* = \sum_{i=1}^d p_E V_i A' V_i^* p_E \\ &= p_E A' \sum_{i=1}^d V_i V_i^* p_E = p_E A' p_E = X \end{aligned}$$

So  $X$  is a fixed point of  $\Phi_E$  and clearly  $\kappa(X) = x$ . We conclude that  $\kappa$  is also surjective. The fact that  $\kappa$  is isometric on selfadjoint elements is also a consequence of Lemma 4.6.4.

On the other hand, if the lifting  $\underline{E}$  of  $\underline{C}$  is not subisometric then the mid  $\underline{V}^C$  of  $\underline{C}$  is embedded on a proper reducing subspace  $\hat{\mathcal{H}}_C$  into the space  $\hat{\mathcal{H}}_E$  of the mid  $\underline{V}^E$  of  $\underline{E}$ . Then  $\mathbf{1}_{\hat{\mathcal{H}}_E}$  and  $p_{\hat{\mathcal{H}}_C}$  are two different fixed points of  $\Phi_{V^E}$ . By Lemma 4.6.4 the map  $\hat{X} \mapsto p_E \hat{X} p_E$  maps them into different fixed points of  $\Phi_E$ :  $p_E \mathbf{1}_{\hat{\mathcal{H}}_E} p_E = p_E \neq p_E p_{\hat{\mathcal{H}}_C} p_E$ . If  $\kappa : X \mapsto p_C X p_C$  from the fixed point set of  $\Phi_E$  into the fixed point set of  $\Phi_C$  were injective then also  $p_C p_E p_C \neq p_C p_E p_{\hat{\mathcal{H}}_C} p_E p_C$ . But both sides are equal to  $p_C$ . Hence in this case  $\kappa$  is not injective. We have proved (a)  $\Leftrightarrow$  (d).  $\square$

Recall that a unital completely positive map  $\Phi_E$  is called *ergodic* if there are no other fixed points than the multiples of the identity. By abuse of language we also call  $\underline{E}$  ergodic in this case (as in [DG07a]).

**Proposition 4.5.5.** *Let  $\underline{E}$  be a coisometric lifting of  $\underline{C}$  by  $\underline{A}$ . Then  $\underline{E}$  is ergodic if and only if  $\underline{C}$  is ergodic and  $\underline{A}$  is  $*$ -stable.*

*Proof.* If  $\underline{A}$  is  $*$ -stable then use the equivalence (b)  $\Leftrightarrow$  (d) in Theorem 4.5.4 and infer from  $\underline{C}$  ergodic that also  $\underline{E}$  is ergodic. Further note that, because  $\underline{E}$  is coisometric, we always have

$$\Phi_E \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \underline{B}\underline{B}^* \end{pmatrix} \geq \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}$$

We say that  $p_C \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}$  is an increasing projection for  $\Phi_E$ . Hence  $(\Phi_E)^n(p_C)$  increases to a SOT-limit which clearly is a fixed point of  $\Phi_E$ .

Now let  $\underline{E}$  be ergodic. Then all fixed points are multiples of  $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$  and because the left upper corner of  $(\Phi_E)^n(p_C)$  is always  $\mathbf{1}$  we have

$(\Phi_E)^n(p_C) \rightarrow \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ . We have verified Theorem 4.5.4(c) and now Theorem 4.5.4(d) and (b) show that  $\underline{C}$  is ergodic and that  $\underline{A}$  is  $*$ -stable.  $\square$

This generalizes Proposition 2.3 in [DG07a] where  $\mathcal{H}_C$  is one dimensional and hence  $\underline{C}$  ergodic is automatically fulfilled.

The following provides an interesting example for the liftings considered above. Let  $\Phi_E : \mathcal{B}(\mathcal{H}_E) \rightarrow \mathcal{B}(\mathcal{H}_E)$  be any (normal) unital completely positive map and let  $\psi$  be a normal invariant state, i.e.,  $\psi \circ \Phi_E = \psi$ . Define  $\mathcal{H}_C$  to be the support of  $\psi$  (cf. [Ta01]) and let  $\mathcal{H}_A$  be the orthogonal complement, so  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$ . Then  $\underline{E} = (E_1, \dots, E_d)$  is a coisometric lifting of  $\underline{C} = (C_1, \dots, C_d)$  if we define  $C_i^* := E_i^*|_{\mathcal{H}_C}$  for  $i = 1, \dots, d$ . In fact  $p_C E_i^* p_C = E_i^* p_C$  for all  $i$  by Lemma 6.1 of [BJKW00]. Note that the compression  $\Phi_C$  has a faithful normal invariant state, the restriction  $\psi_C$  of  $\psi$  to  $\mathcal{B}(\mathcal{H}_C)$ . Conversely we can start with  $\Phi_C$  and a faithful invariant state  $\psi_C$  and construct liftings  $\Phi_E$ . They have normal invariant states given by  $\psi(X) := \psi_C(p_C X p_C)$ . From Proposition 4.2.3 and Theorem 4.5.4 we conclude

**Corollary 4.5.6.** *Let  $\Phi_C : \mathcal{B}(\mathcal{H}_C) \rightarrow \mathcal{B}(\mathcal{H}_C)$  be a (normal) unital completely positive map with a faithful normal invariant state  $\psi_C$ . Then we have a one-to-one correspondence between*

- (a) (normal) unital completely positive maps  $\Phi_E : \mathcal{B}(\mathcal{H}_E) \rightarrow \mathcal{B}(\mathcal{H}_E)$  with normal invariant state  $\psi$  such that the support of  $\psi$  is  $\mathcal{H}_C$  and  $\psi|_{\mathcal{B}(\mathcal{H}_C)} = \psi_C$ , compression of  $\Phi_E$  is  $\Phi_C$  and  $(\Phi_E)^n(p_C) \rightarrow \mathbf{1}$  ( $n \rightarrow \infty$ , SOT)
- (b)  $*$ -stable  $\underline{A}$  with  $\dim \mathcal{D}_{*,A} \leq \dim \mathcal{D}_C$  together with isometries  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$

*There exist order isomorphisms  $\kappa_E : X \mapsto p_C X p_C$  between the fixed point sets of these maps  $\Phi_E$  and the fixed point set of  $\Phi_C$ .*

In the special case when  $\psi$  is an invariant vector state  $\langle \xi, \cdot \xi \rangle$  of  $\Phi_E$  we have the result that  $\Phi_E$  is ergodic if and only if  $(\Phi_E)^n(p_\xi) \rightarrow \mathbf{1}$  ( $n \rightarrow \infty$ , *SOT*), where  $p_\xi$  is the orthogonal projection onto  $\mathbb{C}\xi$ , cf. [Go04], A.5.2. Hence we obtain a classification of such maps. Here  $\mathcal{D}_C$  is  $(d-1)$ -dimensional. This case has been further investigated in [DG07a].

Corollary 4.5.6 is useful because many techniques only apply to completely positive maps with faithful invariant states, cf. [Kü03]. It enables us to transfer information from the faithful to the non-faithful setting. For example, it is known that in the case of a faithful normal invariant state the fixed point set is an algebra (cf. [Ch74, FNW94, BJKW00]). Now  $\kappa$  is an order isomorphism but it is not in general multiplicative. In fact, there are examples of completely positive maps with a normal invariant non-faithful state where the fixed point set is not an algebra (cf. [Ar69, Ar72, BJKW00]). If Corollary 4.5.6 applies we can think of it as an (order isomorphic) deformation of an algebra.

## 4.6 Appendix

In Section 4.5 we needed a commutant lifting theorem (Theorem 5.1 of [BJKW00]) which says that the fixed point set of a normal unital completely positive map is in one-to-one correspondence with the commutant of the Cuntz algebra representation generated by the mid. Below we give a variant of the proof which is based on a Radon-Nikodym result for completely positive maps by W.Arveson. This is a good way to think about it and it supports the understanding of the other arguments in the main text.

**Lemma 4.6.1.** [Ar69], Theorem 1.4.2

If  $\Psi$  is a completely positive map from a  $C^*$ -algebra  $\mathcal{B}$  to  $\mathcal{B}(\mathcal{H})$ , with  $\mathcal{H}$  a Hilbert space, then there exists an affine order isomorphism of the partially ordered set of operators  $\{A' \in \pi(\mathcal{B})' : 0 \leq A' \leq \mathbf{1}\}$  onto  $[0, \Psi]$ . Here  $\pi$  is the minimal Stinespring representation of  $\mathcal{B}$  associated to  $\Psi$  and  $[0, \Psi]$  is the order interval containing all completely positive maps  $\Phi : \mathcal{B} \rightarrow \mathcal{B}(\mathcal{H})$  with  $0 \leq \Phi \leq \Psi$ . The order relation for completely positive maps used here is  $\Phi \leq \Psi$  if  $\Psi - \Phi$  is completely positive.

Explicitly, if  $\Psi(x) = W^* \pi(x) W$  is the minimal Stinespring representation of  $\Psi$  then  $A' \in \pi(\mathcal{B})'$  corresponds to  $\Phi = W^* A' \pi(x) W$ .

**Lemma 4.6.2.** [BJKW00], Corollary 2.4; [Po03], Theorem 2.1

If  $0 \leq D \leq \mathbf{1}$  is a fixed point of the (normal unital completely positive) map  $\Phi_R(\cdot) = \sum_1^d R_i \cdot R_i^*$  on  $\mathcal{B}(\mathcal{H})$  then there exists a completely positive map  $\Psi_D : \mathcal{O}_d \rightarrow \mathcal{B}(\mathcal{H})$ ,  $V_\alpha V_\beta^* \mapsto R_\alpha D R_\beta^*$ . Here  $\alpha, \beta \in \tilde{\Lambda}$  and  $\mathcal{O}_d$  is the Cuntz algebra generated by the  $V_i$ , where  $\underline{V} = (V_1, \dots, V_d)$  is a mid of  $\underline{R} = (R_1, \dots, R_d)$ .

Using notation from the previous lemmas we get

**Lemma 4.6.3.** There exists an affine order isomorphism  $D \mapsto \Psi_D$  between

$$\{0 \leq D \leq \mathbf{1} : D \text{ is a fixed point of } \Phi_R(\cdot) = \sum_1^d R_i \cdot R_i^* \text{ on } \mathcal{B}(\mathcal{H})\}$$

and  $[0, \Psi_1]$ , where  $\Psi_1$  is the completely positive map described in Lemma 4.6.2 with  $D = \mathbf{1}$ , i.e.,  $\Psi_1 : \mathcal{O}_d \rightarrow \mathcal{B}(\mathcal{H})$ ,  $V_\alpha V_\beta^* \mapsto R_\alpha R_\beta^*$ .

*Proof.* From  $\Psi_1 = \Psi_D + \Psi_{1-D}$  we see that  $\Psi_D \in [0, \Psi_1]$  for all fixed points  $0 \leq D \leq \mathbf{1}$  of  $\Phi_R$ . On the other hand, if  $\Phi \in [0, \Psi_1]$  then by Lemma 4.6.1 with  $\mathcal{B} = \mathcal{O}_d$  there exists  $A' \in \pi(\mathcal{B})'$  with  $0 \leq A' \leq \mathbf{1}$  such that  $\Phi(x) = W^* A' \pi(x) W$ , where  $\Psi_1(x) = W^* \pi(x) W$  is a minimal Stinespring

representation. Using that  $(V_1, \dots, V_d)$  is a mid of  $\underline{R} = (R_1, \dots, R_d)$  it is easily checked that  $\Psi_1(x) = p\pi(x)p$  is such a minimal Stinespring representation if  $\pi$  is the Cuntz algebra representation generated by  $(V_1, \dots, V_d)$  and  $p$  is the projection onto the space  $\mathcal{H}$ . (In  $p\pi(x)p$  the  $p$  on the right hand side should be interpreted as the embedding of  $\mathcal{H}$  into the dilation space.)

Hence if  $x = V_\alpha V_\beta^*$  then we obtain

$$\Phi(V_\alpha V_\beta^*) = pA'V_\alpha V_\beta^*p = pV_\alpha A'V_\beta^*p = pV_\alpha pA'pV_\beta^*p = R_\alpha pA'pR_\beta^*.$$

We conclude that  $\Phi = \Psi_D$  with  $D := pA'p$ . Clearly  $0 \leq D \leq \mathbf{1}$  and  $D$  is a fixed point of  $\Phi_R$  (because  $\underline{V}$  is a coisometric lifting of  $\underline{R}$ , i.e.,  $\sum_{i=1}^d V_i V_i^* = \mathbf{1}$  and  $R_i p = pV_i$  for all  $i$ ). The correspondence is bijective ( $\Psi_D(\mathbf{1}) = D$ ) and it clearly respects the order.  $\square$

**Lemma 4.6.4.** *[BJKW00], Theorem 5.1*

*There is an affine order isomorphism between  $\{0 \leq D \leq \mathbf{1} : D \text{ is a fixed point of } \Phi_R(\cdot) = \sum_{i=1}^d R_i \cdot R_i^* \text{ on } \mathcal{B}(\mathcal{H})\}$  and  $\{A' \in \pi(\mathcal{O}_d)' : 0 \leq A' \leq \mathbf{1}\}$ , where  $\pi$  is the Cuntz algebra representation generated by the mid  $\underline{V} = (V_1, \dots, V_d)$  of  $\underline{R} = (R_1, \dots, R_d)$ . It is given by  $A' \mapsto pA'p$ , where  $p$  is the projection onto the space  $\mathcal{H}$ . The isomorphism is isometric on the selfadjoint parts.*

*Proof.* For the first part we only have to add to the arguments in the proof of Lemma 4.6.3 the reminder that by Lemma 4.6.1 the correspondence between  $\{A' \in \pi(\mathcal{O}_d)' : 0 \leq A' \leq \mathbf{1}\}$  and  $[0, \Psi_1]$  is a bijection. As pointed out in [BJKW00], Section 4, it is isometric on the selfadjoint parts because  $\mathbf{1}$  is mapped to  $\mathbf{1}$  (identities on different Hilbert spaces) and for selfadjoint elements  $y$  we have  $\|y\| = \inf\{\alpha > 0 : -\alpha\mathbf{1} \leq y \leq \alpha\mathbf{1}\}$ .  $\square$



# Chapter 5

## Constrained Liftings

In the preceding chapter we have mainly exploited the dilation theory of (noncommutative) row contractions. There are some more important types of dilation theories. They are for row contractions satisfying some constraints. Formally:

**Definition** Assume  $\underline{T}$  to be a  $d$ -tuple of bounded operators on a common Hilbert space  $\mathcal{H}$  and  $\{p_\eta(\underline{z})\}_{\eta \in J}$  a finite set of polynomials in  $d$  noncommuting variables with index set  $J$ . Then  $\underline{T}$  is said to be  $J$ -constrained if

$$p_\eta(\underline{T}) = 0 \quad \text{for } \eta \in J.$$

The corresponding dilations are called constrained dilations (defined in Section 5.1). In [BBD04], [De07a] and [BDZ06] the question how the mid is related to the constrained dilation was addressed. Here we carry out this study further in Section 5.1, especially when the row contraction is defined on a finite dimensional Hilbert space. There is a generalization of Beurling's Theorem for Fock spaces by Popescu [Po05] in constrained case. A complete invariant for constrained liftings by c.n.c. is obtained in Section 5.2 motivated by this Beurling type result. These invariants are at

times more interesting than the ones in preceding chapters. For instance in the commuting case this invariant is a  $H^\infty$  function in the sense of multivariable complex analysis. Similar analysis to classify constrained c.n.c. row contractions can be found in [Ar03],[Po07],[BT07],[BES05], etc.

Beginning with a row contraction  $\underline{R} = (R_1, \dots, R_d)$  on a Hilbert space  $\mathcal{L}$ , define

$$\begin{aligned} \mathcal{C}(\underline{R}) : &= \{ \mathcal{M} \subset \mathcal{L} : \mathcal{M} \text{ is an invariant subspace for each } R_i^*, \\ &\quad (p_\eta(\underline{R}))^* h = 0, \forall h \in \mathcal{M}, \forall \eta \in J \}. \end{aligned}$$

$\mathcal{C}(\underline{R})$  is a complete lattice with respect to arbitrary intersections and span closures of arbitrary unions. The maximal element is called *maximal  $J$ -constrained subspace* and is denoted by  $\mathcal{L}^J(\underline{R})$  or  $\mathcal{L}^J$ . The row contraction  $\underline{R}^J = (R_1^J, \dots, R_d^J)$  obtained by compressing  $\underline{R}$  to  $\mathcal{L}^J$  is called *maximal  $J$ -constrained piece*. Clearly the maximal  $J$ -constrained piece is  $J$ -constrained row contraction. The block form of  $\underline{R}$  in terms of the maximal constrained piece is:  $R_i = \begin{pmatrix} R_i^J & 0 \\ \tilde{R}_i & R_i^N \end{pmatrix}$  where  $\underline{R}^N$  is the compression of  $\underline{R}$  to the orthogonal complement of  $\mathcal{L}^J$ .

The three special sets of polynomials  $\{p_\eta\}_{\eta \in J}$  inducing constraints are:

1.  $p_{i,j}(\underline{z}) = z_i z_j - z_j z_i$  for  $(i, j) \in \{1, 2, \dots, d\}^2$  are associated with commuting case.
2.  $p_{i,j}(\underline{z}) = z_i z_j - q_{ji} z_j z_i$  for  $(i, j) \in \{1, 2, \dots, d\}^2$  where  $|q_{ij}| = 1$  and  $q_{ij} = q_{ji}^{-1}$ .
3.  $p_{i,j}(\underline{z}) = z_i z_j - a_{ij} z_i z_j$  for  $(i, j) \in \{1, 2, \dots, d\}^2$  where  $A = (a_{ij})_{d \times d}$  is a 0–1-matrix and each row and column has at least one non zero entry.

If the given set of polynomials is a combination of the above three, we call such cases as  $J'$ -constrained. The  $J$  in the above stated commuting case is denoted by  $J^s$ . For  $J'$ -constrained row contractions of the above mentioned type 1, 2 and 3, the constrained dilations are called standard commuting dilation, q-commuting dilation and Cuntz-Krieger dilation respectively. Note that the type 2 includes the type 1 case.

We recall the definition of Cuntz-Krieger algebras  $\mathcal{O}_A$ . Let  $A = (a_{ij})_{d \times d}$  be a square 0 – 1-matrix i.e.  $a_{ij} \in \{0, 1\}$  and such that each row and column has at least one non-zero entry.

**Definition**  $\mathcal{O}_A$  is the universal  $C^*$ -algebra generated by  $d$  partial isometries  $s_1, \dots, s_d$  with orthogonal ranges satisfying

$$\begin{aligned} s_i^* s_i &= \sum_{j=1}^d a_{ij} s_j s_j^* \\ \mathbf{1} &= \sum_{i=1}^d s_i s_i^*. \end{aligned} \tag{5.1}$$

## 5.1 Minimal constrained dilation

In this section we assume  $d < \infty$ . Lemma 5.1.3 is the only exception where this assumption is not needed. Consider the maximal  $J'$ -constrained piece  $\underline{L}^{J'}$  of the creation operators  $\underline{L}$  on the Fock space  $\Gamma(\mathbb{C}^d)$  or  $\Gamma$ . It is not difficult to show that the unital  $C^*$ -algebras generated by  $\underline{L}^{J'}$  satisfy

$$C^*(\underline{L}^{J'}) = \overline{\text{span}}\{L_\alpha^{J'}(L_\beta^{J'})^* : \alpha, \beta \in \tilde{\Lambda}\}.$$

Using this and the Poisson kernel (2.6) one can show further for any given row contraction  $\underline{T} = (T_1, \dots, T_d)$  on  $\mathcal{H}$  that there exist unique unital completely positive map  $\psi : C^*(\underline{L}) \rightarrow B(\mathcal{H})$  satisfying:

$$\psi(L_\alpha(L_\beta)^*) = T_\alpha(T_\beta)^*.$$

If  $\underline{T}$  is  $J'$ -constrained, then there also exist unique unital completely positive map  $\phi : C^*(\underline{L}^{J'}) \rightarrow B(\mathcal{H})$  with

$$\phi(L_\alpha^{J'}(L_\beta^{J'})^*) = T_\alpha(T_\beta)^*, \quad \alpha, \beta \in \tilde{\Lambda}. \quad (5.2)$$

We can use a minimal Stinespring dilation  $\pi_1 : C^*(\underline{L}^{J'}) \rightarrow B(\mathcal{H}_1)$  of  $\psi$  such that

$$\phi(X) = P_{\mathcal{H}}\pi_1(X)|_{\mathcal{H}} \quad \forall X \in C^*(\underline{L}^{J'})$$

and  $\overline{\text{span}}\{\pi_1(X)h : X \in C^*(\underline{L}^{J'}), h \in \mathcal{H}\} = \mathcal{H}_1$ . The tuple  $\tilde{\underline{S}} = (\tilde{S}_1, \dots, \tilde{S}_d)$  where  $\tilde{S}_i = \pi_1(L_i^{J'})$ , is the *minimal  $J'$ -constrained dilation* of  $\underline{T}$  which is unique up to unitary equivalence.

We recall a result from [BBD04], [De07a] and [BDZ06] (cf. Appendix of this chapter):

**Theorem 5.1.1.** *Let  $\underline{T}$  be a  $J'$ -constrained row contraction on  $\mathcal{H}$ . Then  $\underline{V}^{J'}$  (i.e., the maximal  $J'$ -constrained piece of  $\underline{V}$ ) is the minimal  $J'$ -constrained dilation of  $\underline{T}$ .*

From the easy observation that the maximal constrained subspace of  $\underline{T}$  is a  $\underline{V}$ -coinvariant subspace we have  $\underline{V}$  to be an isometric lifting of  $\underline{T}^J$ . Therefore using the previous Theorem we get that the compression of  $\underline{V}$  to a  $\underline{V}^{J'}$ -coinvariant subspace of  $\hat{\mathcal{H}}^{J'}$  is the constrained dilation of  $\underline{T}^{J'}$ . It is natural to ask if  $\underline{V}^{J'}$  is the minimal constrained dilation of  $\underline{T}^{J'}$ . For  $*$ -stable row contraction a necessary and sufficient condition appears in Theorem 9 of [BBD04]. A version for coisometric case is obtained here in Theorem 5.1.4 and another for finite dimensional Hilbert spaces in Theorem 5.1.7.

We will need the following lemma. Let  $\mathcal{G}$  be the (non-selfadjoint unital) weak operator topology-closed algebra generated by the  $V_i \in B(\hat{\mathcal{H}})$  of the mid  $\underline{V}$  of  $\underline{T}$ .

**Lemma 5.1.2.** (Lemma 3.4 of [DKS01])  $\mathcal{G}[\mathcal{L}]$  reduces  $\mathcal{G}$  if  $\mathcal{L}$  is a  $T_i^*$ -invariant subspace.

**Lemma 5.1.3.** Suppose  $\underline{T}$ , given on a finite dimensional  $\mathcal{H}$ , is coisometric. Let  $\mathcal{M}$  be a subspace of (the dilation space)  $\hat{\mathcal{H}}$  which is invariant for both  $\mathcal{G}$  and  $\mathcal{G}^*$  (i.e., reducing). Denote  $\mathcal{M} \cap \mathcal{H}$  by  $\mathcal{H}_{\mathcal{M}}$ . Then  $\mathcal{M} = \mathcal{G}[\mathcal{H}_{\mathcal{M}}]$ .

*Proof.* Note that  $\mathcal{H}_{\mathcal{M}}$  is invariant with respect to  $T_i^*$  for  $i = 1, \dots, d$  because  $\mathcal{H}$  and  $\mathcal{M}$  are  $V_i^*$ -invariant and  $V_i^*|_{\mathcal{H}} = T_i^*$ . Lemma 5.1.2 shows that  $\mathcal{G}[\mathcal{H}_{\mathcal{M}}]$  reduces  $\mathcal{G}$ . Because  $\mathcal{H}_{\mathcal{M}} \subset \mathcal{M}$  and  $\mathcal{M}$  is  $\mathcal{G}$ -invariant we also have  $\mathcal{G}[\mathcal{H}_{\mathcal{M}}] \subset \mathcal{M}$ . Let us assume that  $\mathcal{H}' = \mathcal{M} \ominus \mathcal{G}[\mathcal{H}_{\mathcal{M}}]$  is non-zero. But Corollary 4.2 of [DKS01] gives that any non-zero  $\mathcal{G}^*$ -invariant subspace intersects  $\mathcal{H}$  nontrivially. Hence  $\mathcal{H}'$  has a non-trivial intersection with  $\mathcal{H}$ . This is a contradiction as  $(\mathcal{M} \ominus \mathcal{G}[\mathcal{H}_{\mathcal{M}}]) \cap \mathcal{H} = \mathcal{H}_{\mathcal{M}} \cap \mathcal{G}[\mathcal{H}_{\mathcal{M}}]^\perp = \{0\}$ . Therefore  $\mathcal{M} = \mathcal{G}[\mathcal{H}_{\mathcal{M}}]$ .  $\square$

Suppose  $\pi$  is a representation on a Hilbert space  $\mathcal{L}$  of the Cuntz algebra  $\mathcal{O}_d$  with generators  $g_1, \dots, g_d$ . The representation  $\pi$  is said to be *spherical* if  $\overline{\text{span}} \{ \pi(g_\alpha) h : h \in \mathcal{L}^s(\pi(g_1), \dots, \pi(g_d)), \alpha \in \tilde{\Lambda} \} = \mathcal{L}$ .

Let us now assume that  $\mathcal{H}$  is finite dimensional. Theorem 19 of [BBD04] states that the mid  $\underline{V}$  on  $\hat{\mathcal{H}}$  of  $\underline{T}$  on  $\mathcal{H}$  can be decomposed as  $\underline{V}^0 \oplus \underline{V}^1$  with respect to the decomposition of  $\hat{\mathcal{H}}$  as  $\hat{\mathcal{H}}^0 \oplus \hat{\mathcal{H}}^1$  into reducing subspaces where  $\underline{V}^0$  is associated to a spherical representation of  $\mathcal{O}_d$  and  $\underline{V}^1$  has trivial maximal commuting piece. Because  $\mathcal{H}$  is finite dimensional, the already mentioned direct integral decomposition ([BBD04], Theorem 18) now tells us that  $\hat{\mathcal{H}}^0$  can be further decomposed into irreducible subspaces as  $\hat{\mathcal{H}}_1^0 \oplus \dots \oplus \hat{\mathcal{H}}_k^0$  for some  $k \in \mathbb{N}$ . Let  $\mathcal{H}_j := \mathcal{H} \cap \hat{\mathcal{H}}_j^0$ . We observe that  $\mathcal{H}_j, j = 1, 2, \dots, k$ , are non-zero disjoint  $T_i^*$ -invariant subspaces for  $i = 1, \dots, d$  and  $\mathcal{G}[\mathcal{H}_j] = \hat{\mathcal{H}}_j^0$  for  $j = 1, \dots, k$  from Lemma 5.1.3. It follows also that the compressions of  $\underline{T}$  to the  $\mathcal{H}_j$ 's are coisometric. But as the restriction of  $\underline{V}$  to  $\hat{\mathcal{H}}_j^0$  is associated to an irreducible and

spherical representation, the related maximal commuting subspace is one dimensional (cf. [BBD04], Theorem 18 and 19) and hence is a minimal  $\mathcal{G}^*$ -invariant subspace for each  $j$ . By Lemma 5.8 of [DKS01] such a minimal  $\mathcal{G}^*$ -invariant subspace is unique, and since the  $\mathcal{H}_j$ 's are  $\mathcal{G}^*$ -invariant subspaces, it follows that the maximal commuting subspace of  $\underline{V}$  on  $\hat{\mathcal{H}}_j^0$  is contained in  $\mathcal{H}_j$ .

Consider the case when the maximal commuting subspace of the mid  $\underline{V}$  of a row contraction  $\underline{T}$  on the Hilbert space  $\mathcal{H}$  is contained in  $\mathcal{H}$ . Proposition 7 of [BBD04] yields that  $\hat{\mathcal{H}}^{J^s} \cap \mathcal{H} = \mathcal{H}^{J^s}$ . So the maximal commuting piece of  $\underline{T}$  is also the maximal commuting piece of  $\underline{V}$  and therefore the standard commuting dilation of itself.

**Theorem 5.1.4.** *Suppose the dimension of  $\mathcal{H}$  is finite and  $\underline{T}$  is a coisometric row contraction on it. Then the maximal commuting subspace of  $\underline{V}$  is contained in  $\mathcal{H}$  and coincides with the maximal commuting subspace of  $\underline{T}$ .*

*Proof.* Let  $\underline{V}$  on  $\hat{\mathcal{H}}$  be decomposed as above. From the arguments above, we obtain that the maximal commuting subspaces of the compressions of  $\underline{V}$  on  $\hat{\mathcal{H}}_j^0$  are contained in  $\mathcal{H}_j$ . The linear span of all these subspaces is in fact  $\hat{\mathcal{H}}^{J^s}(\underline{V})$  and hence is also contained in  $\mathcal{H}$ . The argument for the second assertion has already been given above also.  $\square$

For a lifting  $\underline{E}$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  of  $\underline{C}$  by  $\underline{A}$ , it is evident that when  $\underline{E}$  is  $J$ -constrained, then both  $\underline{C}$  and  $\underline{A}$  are  $J$ -constrained. In case  $\underline{E}$  is a subisometric lifting of a row contraction  $\underline{C}$ , then the unitary equivalence of mids  $\underline{V}^C$  and  $\underline{V}^E$  implies unitary equivalence of the maximal constrained pieces of  $\underline{V}^C$  and  $\underline{V}^E$ . In addition when  $\underline{E}$  is  $J'$ -constrained, by Theorem 5.1.1 we obtain unitary equivalence of their minimal constrained dilations.

**Corollary 5.1.5.** *Let  $\underline{E}$ , on a finite dimensional Hilbert space, be a coisometric lifting of  $\underline{C}$  by a  $*$ -stable row contraction  $\underline{A}$ . Then the maximal*

commuting pieces of  $\underline{E}$  and  $\underline{C}$  coincide.

*Proof.* Here  $\underline{E}$  is a subisometric lifting of  $\underline{C}$ . So maximal commuting pieces of  $\underline{V}^C$  and  $\underline{V}^E$  are unitarily equivalent. By the previous Theorem this means that the maximal commuting pieces of  $\underline{E}$  and  $\underline{C}$  coincide.  $\square$

**Theorem 5.1.6.** *Suppose  $\underline{T}$  is coisometric and  $\underline{T}^N$  is  $*$ -stable. Then the minimal  $J'$ -constrained dilation of a  $\underline{T}^{J'}$  is the maximal constrained piece of the mid  $\underline{T}$ .*

*Proof.* As  $\underline{T}^N$  is  $*$ -stable, by Proposition 4.2.3  $\underline{T}$  is a subisometric lifting of  $\underline{T}^{J'}$ . This causes the maximal  $J'$ -constrained pieces of mids of  $\underline{T}$  and  $\underline{T}^{J'}$  to be unitarily equivalent. Moreover the maximal constrained piece of mid of  $\underline{T}^{J'}$  is the minimal constrained dilation of  $\underline{T}^{J'}$ . Hence the Theorem follows.  $\square$

The maximal commuting piece of any coisometric row contraction  $\underline{T}$  on  $\mathcal{H}$  consists of subnormal operators. This is because the maximal commuting piece of a coisometric row contraction is commuting and coisometric, and so it has a standard commuting dilation consisting of normal operators (cf. [Ar98], Corollary 1 in Section 8). Consequently if  $\mathcal{H}$  is finite dimensional, the maximal commuting piece of  $\underline{T}$  always consists of normal operators.

**Corollary 5.1.7.** *A commuting coisometric row contraction on a finite dimensional Hilbert space cannot be a lifting of another commuting row contraction by a  $*$ -stable tuple.*

*Proof.* Assume that  $\underline{E}$  is a commuting coisometric lifting of a commuting row contraction  $\underline{C}$  on a finite dimensional Hilbert space by  $*$ -stable  $\underline{A}$ . Then again by Proposition 4.2.3  $\underline{E}$  is a subisometric lifting of  $\underline{C}$ . Consequently, standard commuting dilations of  $\underline{E}$  and  $\underline{C}$  are unitarily equivalent.

For normal coisometric row contractions their standard commuting dilations coincide with the row contractions as shown in Theorem 15 [BBD04]. As  $\underline{C}$  consists of normal operators, its standard commuting dilation is equal to itself. In other words  $\underline{C}$  is equal to the standard commuting dilation of  $\underline{E}$  along with the fact that  $\underline{E}$  is a compression of the standard commuting dilation of  $\underline{E}$ . This yields that  $\underline{C} = \underline{E}$ .  $\square$

**Proposition 5.1.8.** *Let  $C$  be a (single) coisometry on a finite dimensional Hilbert space. Let  $E$  be a coisometric lifting of  $C$  by  $A$ . Then  $A$  is coisometric.*

*Proof.* Let

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}.$$

$C$  is infact a unitary. We have  $C^*C = \mathbf{1}$  and  $BC^* = 0$ . This implies  $B$  is zero and hence  $A$  is coisometric.  $\square$

The following is a passing by remark on the ergodic case treated in Chapter 3.

**Remark 5.1.9.** *If the maximal commuting subspace of a coisometric row contraction  $\underline{T}$  is of dimension greater than one then  $\underline{T}$  is not ergodic.*

*Proof.* If  $\underline{T}$  is ergodic then by Theorem 5.1 of [BJKW00] its mid  $\underline{V}$  on  $\hat{\mathcal{H}}$  is associated to an irreducible representation of  $\mathcal{O}_d$  and so Theorem 19 of [BBD04] tells us that  $\underline{V}$  has trivial or one dimensional maximal commuting piece. But as  $\hat{\mathcal{H}}^{J^s} \cap \mathcal{H} = \mathcal{H}^{J^s}$  by Proposition 7 of [BBD04], we finally get that  $\underline{T}$  has trivial or one dimensional maximal commuting piece.  $\square$



## 5.2 Constrained characteristic function

As before let  $\underline{C}$  be a row contraction on  $\mathcal{H}_C$  and  $\underline{E}$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  be a contractive lifting of  $\underline{C}$  with  $B_i = P_{\mathcal{H}_A} E_i P_{\mathcal{H}_C}$ .

For a reduced lifting if we define  $M := P_{\mathcal{K}^\perp} W P_{\mathcal{H}_E} : \mathcal{H}_E \rightarrow \mathcal{H}_C \oplus \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_C$  then  $M$  acts as identity on  $\mathcal{H}_C$  and

$$Mh := \sum_{\alpha} e_{\alpha} \otimes \gamma D_{*,A} A_{\alpha}^* h \quad \text{for } h \in \mathcal{H}_A$$

where  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  is a contraction as defined before. The characteristic function for this reduced lifting is defined as

$$M_{C,E} := P_{\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_C} W|_{\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_E}.$$

In order to have a similar invariant for  $J$ -constrained row contractions, say  $\underline{T}$  on  $\mathcal{H}$ , we consider a dilation  $\underline{S}^T$  of  $\underline{T}$  obtained by a compression of Popescu's realization of mid to  $\mathcal{H} \oplus (\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_T)$ . Explicitly for  $i = 1, \dots, d$

$$S_i^T(h \oplus d) = T_i h \oplus [e_0 \otimes (D_T)_i h + (L_i^J \otimes \mathbf{1})(d)] \quad \text{for } h \in \mathcal{H}, d \in \Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_T.$$

That  $\underline{S}^T$  is a dilation is immediate and we call it *pseudo  $J$ -constrained dilation* of  $\underline{T}$ . Note that  $\underline{S}^T$  does not satisfy constrained relations. In case of  $J'$  this dilation contains the minimal constrained dilation as a corner (w.r.t. a coinvariant subspace). Indeed Theorem 5.1.1 tells us that the maximal  $J'$ -constrained pieces of mids are minimal constrained dilations. The constrained pieces are clearly on spaces  $\mathcal{H} \oplus \mathcal{N}_J$  where  $\mathcal{N}_J$  are some proper coinvariant subspaces of  $(\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_T)$  with respect to the  $\underline{L}^J \otimes \mathbf{1}$ .

A *constrained lifting* of a row contraction is a lifting which as a tuple of operators is  $J$ -constrained. For a constrained lifting  $\underline{E}$  of  $\underline{C}$ , the tuple  $\underline{A}$  is also  $J$ -constrained and therefore by [Po05]  $M\mathcal{H}_A$  is contained in

$\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C$ . We set  $M_0 := M|_{\mathcal{H}_A}$  and  $M_0^J := P_{\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C} M_0$ . Let us denote the restriction of  $W$  to  $\mathcal{H}_E \oplus (\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E)$  by  $\mathbb{W}$ . Because  $M_{C,E}$  is multianalytic and

$$\Gamma_J(\mathbb{C}^d) = \overline{\text{span}}\{L_\alpha^*(L_i L_j - L_j L_i)^* L_\beta^* f = 0 : f \in \Gamma(\mathbb{C}^d); \alpha, \beta \in \tilde{\Lambda}\}$$

(cf. [BDZ06]), we get for  $k \in \Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E$

$$\begin{aligned} [(L_\alpha^*(L_i L_j - L_j L_i)^* L_\beta^*) \otimes \mathbf{1}] M_{C,E} k &= M_{C,E} [(L_\alpha^*(L_i L_j - L_j L_i)^* L_\beta^*) \otimes \mathbf{1}] k \\ &= 0 \quad \text{for } \alpha, \beta \in \tilde{\Lambda}. \end{aligned}$$

Thus the range of  $P_{\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_C} \mathbb{W}|_{\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E}$  is  $\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C$ . We define the *constrained characteristic function* as  $M_{J,C,E} := P_{\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C} \mathbb{W}|_{\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E}$  which is the same as  $M_{C,E}|_{\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E}$ . Let  $d_h^i := D_E(e_i \otimes h)$  for  $h \in \mathcal{H}_E$ . From equations (4.49) and (4.50) it is clear that the symbol  $\theta_{J,C,E}$  of  $M_{J,C,E}$  for  $h \in \mathcal{H}_C$  will be given by

$$\theta_{J,C,E} d_h^i := e_0 \otimes [\mathcal{D}_C(e_i \otimes h) - \gamma D_{*,A} B_i h] - \sum_{|\alpha| \geq 1} L_\alpha^J e_0 \otimes \gamma D_{*,A} A_\alpha^* B_i h$$

and for  $h \in \mathcal{H}_A$  by

$$\begin{aligned} \theta_{J,C,E} d_h^i &:= -e_0 \otimes \gamma D_{*,A} A_i h + \sum_{\alpha} L_i^J L_\alpha^J e_0 \otimes \gamma D_{*,A} A_\alpha^* (\mathbf{1} - A_i^* A_i) h \\ &\quad + \sum_{j \neq i} \sum_{\alpha} L_j^J L_\alpha^J e_0 \otimes \gamma D_{*,A} A_\alpha^* (-A_j^* A_i) h. \end{aligned}$$

It is evident that  $\underline{S}^E$  is unitarily equivalent to  $\underline{S}^C \oplus \underline{Z}$  where  $\underline{Z}$  is defined on some Hilbert space  $\mathcal{K}_1$  and

$$\mathbb{W} S_i^E = (S_i^C \oplus Z_i) \mathbb{W}.$$

Let  $\mathcal{H}_A^1$  be

$$\mathcal{H}_A^1 := \{h \in \mathcal{H}_A : \sum_{|\alpha|=n} \|A_\alpha^* h\|^2 = \|h\|^2 \text{ for all } n \in \mathbb{N}\}$$

as in the last chapter. The equation (4.36) gives us

$$W\mathcal{H}_A^\perp \perp \mathcal{H}_C \oplus (\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_C).$$

Therefore

$$\mathbb{W}\mathcal{H}_A^\perp \perp \mathcal{H}_C \oplus (\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C)$$

and hence  $\mathbb{W}\mathcal{H}_A^\perp \subset \mathcal{K}_1$ . We denote the space on which  $S_i^E$ s are defined by  $\mathbf{K}_J$ . Assuming  $\gamma$  to be resolving and repeating the arguments from the last chapter in this setting, we have

$$\mathbb{W}\mathbf{K}_J \ominus (\mathcal{H}_C \oplus \mathcal{H}_A^\perp) = (\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C) \bigvee \mathbb{W}(\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E).$$

In case of reduced constrained liftings,  $\mathcal{H}_A^\perp$  is trivial and taking  $\Delta_{J,C,E} := (\mathbf{1} - M_{J,C,E}^* M_{J,C,E})^{\frac{1}{2}}$ , we can realise  $\underline{Z}$  on  $\overline{\Delta_{J,C,E}(\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E)}$ . Hence for such liftings  $\underline{S}^C \oplus \underline{Z}$  is a row contraction on

$$\mathbb{W}\mathbf{K}_J := \mathcal{H}_C \oplus (\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C) \oplus \overline{\Delta_{J,C,E}(\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E)}$$

and

$$\mathbb{W}\mathcal{H}_E := \mathbb{W}\mathbf{K}_J \ominus (M_{J,C,E} \oplus \Delta_{J,C,E})(\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E).$$

is coinvariant for  $\underline{S}^C \oplus \underline{Z}$  and their restriction to  $\mathbb{W}\mathcal{H}_E$  gives a copy of  $\underline{E}$ .

**Theorem 5.2.1.** *For a constrained row contraction, two constrained reduced liftings are unitarily equivalent if and only if its constrained characteristic functions coincide.*

The proof follows from Theorem 4.3.7 after noting that unitary equivalence of two liftings say  $\underline{E}$  and  $\hat{\underline{E}}$  of given row contraction  $\underline{C}$  will also mean unitary equivalence of  $\underline{L} \otimes \mathbf{1}_{\mathcal{D}_E}$  and  $\underline{L} \otimes \mathbf{1}_{\mathcal{D}_{\hat{E}}}$ . This implies unitary equivalence of the pseudo constrained dilations.

Suppose  $\underline{E}$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  be such a  $J$ -constrained lifting of  $\underline{C}$  by a c.n.c. row contraction  $\underline{A}$ . We follow the notations of the last Section. It is immediate that

$$\mathbb{W}(M_0^J)^*v = P_{\mathbb{W}\mathcal{H}_E}(v \oplus 0), \quad \text{for } v \in \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_C.$$

Consequently

$$\|(M_0^J)^*v\| = \|P_{\mathbb{W}\mathcal{H}_E}(v \oplus 0)\|, \quad \text{for } v \in \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_C. \quad (5.3)$$

The map  $\phi : \Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_E \rightarrow \mathbb{W}\mathbf{K}_J$  (in fact  $\mathbb{W}\mathbf{K}_J \ominus \mathcal{H}_C$ ) defined by

$$u \mapsto M_{J,C,E}u \oplus \Delta_{J,C,E}u$$

is an isometry and

$$\phi^*(v \oplus 0) = M_{J,C,E}^*v$$

for  $v \in \Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C$ . Suppose  $P_{\mathbb{W}\mathcal{H}_E} \in B(\mathbf{K}_J)$  is the orthogonal projection onto  $\mathbb{W}\mathcal{H}_E$ . Then for  $v \in \Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C$

$$\|v\|^2 = \|P_{\mathbb{W}\mathcal{H}_E}(v \oplus 0)\|^2 + \|\phi\phi^*(v \oplus 0)\|^2 = \|P_{\mathbb{W}\mathcal{H}_E}(v \oplus 0)\|^2 + \|M_{J,C,E}^*v\|^2. \quad (5.4)$$

Comparing equations (5.3) and (5.4) we get

$$M_0^J(M_0^J)^* + M_{J,C,E}M_{J,C,E}^* = \mathbf{1}_{\Gamma_J(\mathbb{C}^d) \otimes \mathcal{D}_C}. \quad (5.5)$$

When  $J = \{0\}$  we obtain

$$M_0(M_0)^* + M_{C,E}M_{C,E}^* = \mathbf{1}_{\Gamma(\mathbb{C}^d) \otimes \mathcal{D}_C}. \quad (5.6)$$

**Remark 5.2.2.** *The above results clearly holds for any contractive liftings which are resolving, i.e., the liftings need not necessarily be reduced.*

## 5.3 Appendix

In Lemma 5.1.1 we refer to a result from [BBD04], [De07a] and [BDZ06]. We illustrate how one can prove it in a special case in this Appendix.

**Theorem 5.3.1.** *Let  $\underline{T} = (T_1, \dots, T_d)$  be a row contraction on a Hilbert space  $\mathcal{H}$  where  $T_i$ 's are mutually commuting. Then the maximal commuting piece of the mid of  $\underline{T}$  is a realization of the standard commuting dilation of  $\underline{T}$ .*

It is known that the maximal  $J^s$ -constrained subspace (defined in Section 5.1) of  $\Gamma(\mathbb{C}^d)$  w.r.t. the standard tuple  $\underline{L}$  of creation operators is the symmetric Fock space  $\Gamma_s(\mathbb{C}^d)$ . Let us denote by  $\underline{S}$  the constrained piece  $\underline{L}^{J^s}$ . To prove the above theorem we consider the map  $\phi : C(\underline{S}) \rightarrow B(\mathcal{H})$  from the Equation (5.2) associated with the standard commuting dilation of  $\underline{T}$  and the dilation  $\pi_1$  of  $\phi$  on a Hilbert space  $\mathcal{H}_1$ . As  $\underline{S}$  is also a row contraction, out of the discussion at the beginning of Section 5.1 it follows that there is a unique unital completely positive map  $\eta : C^*(\underline{L}) \rightarrow C^*(\underline{S})$ , satisfying

$$\eta(L_\alpha(L_\beta)^*) = S_\alpha(S_\beta)^*, \quad \alpha, \beta \in \tilde{\Lambda}.$$

Consider the minimal Stinespring dilation of the composed map  $\pi_1 \circ \eta : C^*(\underline{L}) \rightarrow \mathcal{B}(\mathcal{H}_1)$ . We obtain a Hilbert space  $\mathcal{H}_2$  containing  $\mathcal{H}_1$  and a unital  $*$ -homomorphism  $\pi_2 : C^*(\underline{L}) \rightarrow \mathcal{B}(\mathcal{H}_2)$ , such that

$$\pi_1 \circ \eta(X) = P_{\mathcal{H}_1} \pi_2(X)|_{\mathcal{H}_1}, \quad X \in C^*(\underline{L}),$$

and  $\overline{\text{span}} \{ \pi_2(X)h : X \in C^*(\underline{L}), h \in \mathcal{H}_1 \} = \mathcal{H}_2$ . In the following commuting diagram

$$\begin{array}{ccccc}
& & & & \mathcal{B}(\mathcal{H}_2) \\
& & & & \downarrow \\
& & & & \mathcal{B}(\mathcal{H}_1) \\
& & & & \downarrow \\
& & & & \mathcal{B}(\mathcal{H}) \\
& \nearrow \pi_2 & & \nearrow \pi_1 & \\
C^*(\underline{L}) & \xrightarrow[\eta]{} & C^*(\underline{S}) & \xrightarrow[\phi]{} & 
\end{array}$$

horizontal arrows are unital completely positive maps, down arrows are compressions and diagonal arrows are minimal Stinespring dilations.

Taking  $\hat{\underline{V}} = (\hat{V}_1, \dots, \hat{V}_d) = (\pi_2(L_1), \dots, \pi_2(L_d))$ , we first show that  $\tilde{\underline{S}} = (\pi_1(S_1), \dots, \pi_1(S_d))$  is the maximal commuting piece of  $\hat{\underline{V}}$ . Then we prove that  $\hat{\underline{V}}$  is the mid of  $\underline{T}$ . The last statement follows if we can show that  $\pi_2$  is a minimal dilation of  $\phi \circ \eta$  because the minimal Stinespring dilation is unique up to unitary equivalence. For this we would need the a lemma (cf. Theorem 15 in [BBD04]).

**Definition 5.3.2.** A  $d$ -tuple  $\underline{T} = (T_1, \dots, T_d)$  of operators on a Hilbert space  $\mathcal{H}$  is called a spherical unitary if it is commuting, each  $T_i$  is normal, and  $T_1 T_1^* + \dots + T_d T_d^* = \mathbf{1}$ .

**Lemma 5.3.3.** Let  $\underline{T}$  be a spherical unitary on a Hilbert space  $\mathcal{H}$ . Then the maximal commuting piece of the mid of  $\underline{T}$  is  $\underline{T}$ .

*Proof of Theorem 5.3.1:* As  $C^*(\underline{S})$  contains the ideal of all compact operators by standard  $C^*$ -algebra theory we have a direct sum decomposition of  $\pi_1$  as follows. Take  $\mathcal{H}_1 = \mathcal{H}_{1C} \oplus \mathcal{H}_{1N}$  where  $\mathcal{H}_{1C} = \overline{\text{span}}\{\pi_1(X)h : h \in \mathcal{H}, X \in C^*(\underline{S}) \text{ and } X \text{ is compact}\}$  and  $\mathcal{H}_{1N} = \mathcal{H}_1 \ominus \mathcal{H}_{1C}$ , Clearly  $\mathcal{H}_{1C}$  is a reducing subspace for  $\pi_1$ . Therefore

$$\pi_1(X) = \begin{pmatrix} \pi_{1C}(X) & \\ & \pi_{1N}(X) \end{pmatrix}$$

that is,  $\pi_1 = \pi_{1C} \oplus \pi_{1N}$  where  $\pi_{1C}(X) = P_{\mathcal{H}_{1C}}\pi_1(X)P_{\mathcal{H}_{1C}}$ ,  $\pi_{1N}(X) = P_{\mathcal{H}_{1N}}\pi_1(X)P_{\mathcal{H}_{1N}}$ .  $\pi_{1C}(X)$  is just the identity representation with some multiplicity as remarked in [Ar98]. In other words  $\mathcal{H}_{1C}$  can be factored as  $\mathcal{H}_{1C} = \Gamma_s(\mathbb{C}^d) \otimes \mathcal{D}_{*,T}$ , such that  $\pi_{1C}(X) = X \otimes \mathbf{1}$ . Also  $\pi_{1N}(X) = 0$  for compact  $X$ . Therefore, taking  $Z_i = \pi_{1N}(S_i)$ ,  $\underline{Z} = (Z_1, \dots, Z_d)$  is a spherical unitary.

Now  $\pi_1 \circ \eta = (\pi_{1C} \circ \eta) \oplus (\pi_{1N} \circ \eta)$  and the minimal Stinespring dilation of a direct sum of two completely positive maps is the direct sum of minimal Stinespring dilations. So  $\mathcal{H}_2$  decomposes as  $\mathcal{H}_2 = \mathcal{H}_{2C} \oplus \mathcal{H}_{2N}$ , where  $\mathcal{H}_{2C}, \mathcal{H}_{2N}$  are orthogonal reducing subspaces of  $\pi_2$ , such that  $\pi_2$  also decomposes, say  $\pi_2 = \pi_{2C} \oplus \pi_{2N}$ , with

$$\pi_{1C} \circ \eta(X) = P_{\mathcal{H}_{1C}}\pi_{2C}(X)|_{\mathcal{H}_{1C}}, \quad \pi_{1N} \circ \eta(X) = P_{\mathcal{H}_{1N}}\pi_{2N}(X)|_{\mathcal{H}_{1N}},$$

for  $X \in C^*(\underline{L})$  with  $\mathcal{H}_{2C} = \overline{\text{span}} \{ \pi_{2C}(X)h : X \in C^*(\underline{L}), h \in \mathcal{H}_{1C} \}$  and  $\mathcal{H}_{2N} = \overline{\text{span}} \{ \pi_{2N}(X)h : X \in C^*(\underline{L}), h \in \mathcal{H}_{1N} \}$ . It is also not difficult to see that  $\mathcal{H}_{2C} = \overline{\text{span}} \{ \pi_{2C}(X)h : X \in C^*(\underline{L}), X \text{ compact}, h \in \mathcal{H}_{1C} \}$  and hence  $\mathcal{H}_{2C}$  factors as  $\mathcal{H}_{2C} = \Gamma(\mathbb{C}^d) \otimes \mathcal{D}_{*,T}$  with  $\pi_{2C}(V_i) = V_i \otimes \mathbf{1}$ . Also  $(\pi_{2N}(L_1), \dots, \pi_{2N}(L_d))$  is a mid of spherical unitary  $(Z_1, \dots, Z_d)$ . Hence by Lemma 5.3.3 and some easy observations we get that  $(\pi_1(S_1), \dots, \pi_1(S_d))$  acting on  $\mathcal{H}_1$  is the maximal commuting piece of  $(\pi_2(L_1), \dots, \pi_2(L_d))$ .

Next we show that  $\pi_2$  is the minimal Stinespring dilation of  $\phi \circ \eta$ . Assume that this is not true. Then we get a reducing subspace  $\mathcal{H}_{20}$  for  $\pi_2$  by taking  $\mathcal{H}_{20} = \overline{\text{span}} \{ \pi_2(X)h : X \in C^*(\underline{L}), h \in \mathcal{H} \}$ . Take  $\mathcal{H}_{21} = \mathcal{H}_2 \ominus \mathcal{H}_{20}$  and correspondingly decompose  $\pi_2$  as  $\pi_2 = \pi_{20} \oplus \pi_{21}$ ,

$$\pi_2(X) = \begin{pmatrix} \pi_{20}(X) & \\ & \pi_{21}(X) \end{pmatrix}$$

Note that we already have  $\mathcal{H} \subseteq \mathcal{H}_{20}$ . We claim that  $\mathcal{H}_2 \subseteq \mathcal{H}_{20}$ . Firstly, as  $\mathcal{H}_1$  is the space where the maximal commuting piece of  $(\pi_2(L_1), \dots,$

$\pi_2(L_d)) = (\pi_{20}(L_1) \oplus \pi_{21}(L_1), \dots, \pi_{20}(L_d) \oplus \pi_{21}(L_d))$  acts,  $\mathcal{H}_1$  decomposes as  $\mathcal{H}_1 = \mathcal{H}_{10} \oplus \mathcal{H}_{11}$  for some subspaces  $\mathcal{H}_{10} \subseteq \mathcal{H}_{20}$ , and  $\mathcal{H}_{11} \subseteq \mathcal{H}_{21}$ . So for  $X \in C^*(\underline{L})$ ,  $P_{\mathcal{H}_1}\pi_2(X)P_{\mathcal{H}_1}$ , has the form (see the diagram)

$$P_{\mathcal{H}_1}\pi_2(X)P_{\mathcal{H}_1} = \begin{pmatrix} \pi_{10} \circ \eta(X) & 0 & & \\ & 0 & 0 & \\ & & \pi_{11} \circ \eta(X) & 0 \\ & & 0 & 0 \end{pmatrix}$$

where  $\pi_{10}, \pi_{11}$  are compressions of  $\pi_1$  to  $\mathcal{H}_{10}, \mathcal{H}_{11}$  respectively. As the mapping  $\eta$  from  $C^*(\underline{L})$  to  $C^*(\underline{S})$  is clearly surjective, it follows that  $\mathcal{H}_{10}, \mathcal{H}_{11}$  are reducing subspaces for  $\pi_1$ . Now as  $\mathcal{H}$  is contained in  $\mathcal{H}_{20}$ , in view of minimality of  $\pi_1$  as a Stinespring dilation,  $\mathcal{H}_1 \subseteq \mathcal{H}_{20}$ . But then the minimality of  $\pi_2$  shows that  $\mathcal{H}_2 \subseteq \mathcal{H}_{20}$ . So finally we get  $\mathcal{H}_2 = \mathcal{H}_{20}$ .  $\square$



# Chapter 6

## Modules

In this chapter we focus on two module structures arising in operator algebras. By continuing the analysis from the previous chapters we derive some interesting results for modules of both types.

For commuting row contractions Arveson introduced the concept of Hilbert module in [Ar03] where he also gave some geometric invariants for these modules. These notions were extended to the noncommutative case by Kribs ([Kr01]) and Popescu ([Po02]) independently. We obtain how these invariants of Hilbert modules can be expressed in terms of our characteristic functions in Section 6.1 (cf. [Po05]). The subsequent section is devoted to computing some examples. In the final section we are concerned with the second module structure called Hilbert  $C^*$ -module. We generalize part of our theory developed in chapter 4 for liftings of covariant representations of such modules.

### 6.1 Invariants of Hilbert modules

Let  $\underline{T} = (T_1, \dots, T_d)$  be a mutually commuting  $d$ -tuple of operators on  $\mathcal{H}$ . A *Hilbert module* structure on  $\mathcal{H}$  over the algebra of polynomials in

$d$ -variables  $\mathcal{P} = \mathbb{C}[z_1, \dots, z_d]$  is obtained by setting

$$g.h := g(T_1, \dots, T_d)h, \quad \text{for } g \in \mathcal{P}, h \in \mathcal{H}_E.$$

This Hilbert module is contractive in the sense that:

$$\|z_1.h_1 + \dots + z_d.h_d\|^2 \leq \|h_1\|^2 + \dots + \|h_d\|^2, \quad h_1, \dots, h_d \in \mathcal{H}.$$

We caution the reader that this notion of Hilbert module is different from the ones discussed in Sections 2.4 and 6.3.

We will further assume that  $D_{*,T}$  has finite rank. Consider the function  $T : \mathbb{C}^d \rightarrow B(\mathcal{H})$  given by

$$T(z) = \bar{z}_1 T_d + \dots + \bar{z}_d T_d.$$

If  $z$  belongs to the open unit ball  $B_d$  of  $\mathbb{C}^d$ , then it follows immediately that  $\|T\| < 1$  and  $\mathbf{1} - T(z)$  is invertible. For every  $z \in B_d$  let us form an operator  $F(z) \in B(D_{*,T}(\mathcal{H}))$  by putting

$$F(z)h = D_{*,T}((\mathbf{1} - T(z))^*)^{-1}(\mathbf{1} - T(z))^{-1}D_{*,T}h, \quad h \in \mathcal{D}_{*,T}.$$

Arveson in [Ar03] shows that with respect to the natural rotation-invariant probability measure  $\sigma$  on  $\partial B_d$ , the limit

$$K_0(z) = \lim_{r \uparrow 1} (1 - r^2) \operatorname{trace} F(rz) = 2 \lim_{r \uparrow 1} \operatorname{trace} F(rz)$$

exists for  $\sigma$ -almost every  $z \in \partial B_d$  and he defined the *curvature invariant* of the Hilbert module as the scalar

$$\operatorname{curv}_s(T) := \int_{\partial B_d} K_0(z) d\sigma(z).$$

Another invariant for Hilbert modules is the Euler characteristic  $\chi_s(T)$ . For defining it we need to consider the submodule of  $\mathcal{H}$ :

$$\mathcal{M}_T = \operatorname{span}\{g.h : g \in \mathcal{P}, h \in \mathcal{D}_{*,T}\}.$$

If  $r = \dim \mathcal{D}_{*,T}$  and  $k_1, \dots, k_r$  is a basis for  $\mathcal{D}_{*,T}$ , then

$$\mathcal{M}_T = \{f_1.k_1 + \dots + f_r.k_r : f_i \in \mathcal{P}\}.$$

Thus  $\mathcal{M}_T$  is finitely generated. It follows from Hilbert's syzygy theorem (cf. Theorem 182 of [Ka70]) that  $\mathcal{M}_T$  has finite free resolution, i.e., there is an exact sequence of  $\mathcal{P}$ -modules

$$0 \rightarrow F_n \rightarrow \dots \rightarrow F_2 \rightarrow F_1 \rightarrow \mathcal{M}_T \rightarrow 0$$

where  $F_k$  is a free module of finite rank  $\beta_k$ . The numbers  $\beta_k$  are called *Betti numbers*. The *Euler characteristic* is defined as

$$\chi_s(T) = \sum_{k=1}^n (-1)^{k+1} \beta_k$$

and is independent of the choice of the finite free resolution.

In the above quoted article of Arveson it is further shown that:

$$\begin{aligned} \text{curv}_s T &:= (d-1)! \lim_{n \rightarrow \infty} \frac{\text{trace}[K_{J^s}^*(T)(Q_{\leq n} \otimes \mathbf{1}_{\mathcal{D}_{*,T}})K_{J^s}(T)]}{d^n}, \\ \chi_s(T) &:= d! \lim_{n \rightarrow \infty} \frac{\text{rank}[K_{J^s}^*(T)(Q_{\leq n} \otimes \mathbf{1}_{\mathcal{D}_{*,T}})K_{J^s}(T)]}{n^d}, \end{aligned}$$

where  $Q_{\leq n}$  is the orthogonal projection of the *symmetric Fock space*  $\Gamma_s(\mathbb{C}^d)$  onto  $P_{\Gamma_s(\mathbb{C}^d)}(\mathbb{C} \oplus (\mathbb{C}^d)^{\otimes 2} \oplus \dots \oplus (\mathbb{C}^d)^{\otimes n})$  and  $K_{J^s}(T) := P_{\Gamma_s(\mathbb{C}^d) \otimes \mathcal{D}_{*,T}} K(T)$  is obtained by projecting the Poisson kernel  $K(T)$  on  $\Gamma_s(\mathbb{C}^d) \otimes \mathcal{D}_{*,T}$ . (The Poisson kernel is denoted by  $C$  in Equation (2.6). We are using a different notation here for convenience).

Take  $\mathbb{F}_d^+$  to be the free semigroup with  $d$  generators  $f_1, \dots, f_d$  and denote the corresponding complex free semigroup algebra with  $\mathbb{C}\mathbb{F}_d^+$ . In an analogous way as above, to any row contraction (not necessarily commuting)  $\underline{T} = (T_1, \dots, T_d)$  on a Hilbert space  $\mathcal{H}$ , a contractive *Hilbert module*

over  $\mathbb{CF}_d^+$  (cf. [Po02]) can be associated through

$$g.h := g(T_1, \dots, T_d)h, \quad \text{for } g \in \mathbb{CF}_d^+, h \in \mathcal{H}.$$

The contractivity of the module now means

$$\|f_1.h_1 + \dots + f_d.h_d\|^2 \leq \|h_1\|^2 + \dots + \|h_d\|^2, \quad \text{for } h_1, \dots, h_d \in \mathcal{H}.$$

We recall the definition of the curvature invariant  $\text{curv}T$  and Euler characteristic  $\chi(T)$  of Hilbert modules introduced by Kribs and Popescu (cf. [Kr01], [Po02]):

$$\begin{aligned} \text{curv}T &:= \lim_{n \rightarrow \infty} \frac{\text{trace}[K^*(T)(P_{\leq n} \otimes \mathbf{1}_{\mathcal{D}_{*,T}})K(T)]}{d^n}, \\ \chi(T) &:= \lim_{n \rightarrow \infty} \frac{\text{rank}[K^*(T)(P_{\leq n} \otimes \mathbf{1}_{\mathcal{D}_{*,T}})K(T)]}{1 + d + \dots + d^{n-1}}, \end{aligned}$$

where  $K(T)$  (  $= M_0$  ) is the Poisson kernel of  $\underline{T}$  as before and  $P_{\leq n}$  is the orthogonal projection of  $\Gamma(\mathbb{C}^d)$  onto  $\mathbb{C} \oplus (\mathbb{C}^d)^{\otimes 2} \oplus \dots \oplus (\mathbb{C}^d)^{\otimes n}$ .

These invariants are shown in this section to be related to constrained reduced liftings when  $\gamma$  is an isometry. When  $\underline{E}$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  is such a reduced lifting of  $\underline{C}$  by  $\underline{A}$ , we can get three Hilbert modules namely those associated with  $\underline{C}$ ,  $\underline{A}$  and  $\underline{E}$ . Assume in the sequel that  $\text{rank } D_C$  is finite.

**Theorem 6.1.1.**

$$\begin{aligned} \text{curv}A &= \text{rank}D_C - \lim_{n \rightarrow \infty} \frac{\text{trace}[M_{C,E}M_{C,E}^*(P_{\leq n} \otimes \mathbf{1})]}{d^n}, \\ \chi(A) &= \lim_{n \rightarrow \infty} \frac{\text{rank}[(\mathbf{1} - M_{C,E}M_{C,E}^*)(P_{\leq n} \otimes \mathbf{1})]}{1 + d + \dots + d^{n-1}}. \end{aligned}$$

*Proof.* An easy simplification shows that for a lifting  $\underline{E}$

$$\text{curv}A = \lim_{n \rightarrow \infty} \frac{\text{trace}[M_0^*(P_{\leq n} \otimes \mathbf{1}_{\mathcal{D}_C})M_0]}{d^n}, \quad (6.1)$$

$$\chi(A) = \lim_{n \rightarrow \infty} \frac{\text{rank}[M_0^*(P_{\leq n} \otimes \mathbf{1}_{\mathcal{D}_C})M_0]}{1 + d + \dots + d^{n-1}}. \quad (6.2)$$

Evidently

$$\text{trace} M_0^*(P_{\leq n} \otimes \mathbf{1})M_0 = \text{trace} M_0 M_0^*(P_{\leq n} \otimes \mathbf{1}).$$

Since  $\text{rank } D_C < \infty$  and  $M_0$  is injective on the range of  $M_0^*(P_{\leq n} \otimes \mathbf{1})$  we have

$$\text{rank} M_0^*(P_{\leq n} \otimes \mathbf{1})M_0 = \text{rank} M_0^*(P_{\leq n} \otimes \mathbf{1}) = \text{rank} M_0 M_0^*(P_{\leq n} \otimes \mathbf{1}).$$

Now from equations (5.6), (6.1) and (6.2) the claim follows.  $\square$

Let us consider the case when  $\underline{E}$  is commuting. The symmetric Fock space  $\Gamma_s(\mathbb{C}^d)$  can be identified with the space  $H^2$ , of all analytic functions on the open unit ball  $B_d$ , the reproducing kernel Hilbert space with the kernel  $K_d : B_d \times B_d \rightarrow \mathbb{C}$  defined by

$$K_d(z, w) := \frac{1}{1 - \langle z, w \rangle_{\mathbb{C}^d}} \quad \text{for } z, w \in B_d$$

(see [Ar03]). In this picture  $\underline{L}^{J^s}$  corresponds to the tuple  $(M_{z_1}, \dots, M_{z_d})$  of multiplication operators by the coordinate functions. The constrained characteristic function gets identified with a multiplication operator  $M_{J^s, C, E} : H^2 \otimes \mathcal{D}_E \rightarrow H^2 \otimes \mathcal{D}_C$  with its symbol  $\theta_{J^s, C, E}$  as a  $B(\mathcal{D}_E, \mathcal{D}_C)$ -valued bounded analytic function on  $B_d$ . The  $\text{curv}_s A$  simplifies using similar arguments as for the proof of Theorem 6.1.1 to

$$\text{curv}_s A = (d-1)! \lim_{n \rightarrow \infty} \frac{\text{trace}[(M_0^{J^s})^*(Q_{\leq n} \otimes \mathbf{1}_{\mathcal{D}_{*,A}})M_0^{J^s}]}{d^n},$$

From Theorem A of [Ar03] and equation (5.5) we get

$$\text{curv}_s A = \int_{\partial B_d} \lim_{r \rightarrow 1} \text{trace}[\mathbf{1} - \theta_{J^s, C, E}(r\zeta)\theta_{J^s, C, E}^*(r\zeta)] d\sigma(\zeta)$$

and like in Theorem 6.1.1 we also have the final simplifications

$$\begin{aligned} \text{curv}_s A &= \text{rank} D_C - (d-1)! \lim_{n \rightarrow \infty} \frac{\text{trace}[\theta_{J^s, C, E} \theta_{J^s, C, E}^* (Q_{\leq n} \otimes \mathbf{1})]}{d^n}, \\ \chi_s(A) &= d! \lim_{n \rightarrow \infty} \frac{\text{rank}[(\mathbf{1} - \theta_{J^s, C, E} \theta_{J^s, C, E}^*) (Q_{\leq n} \otimes \mathbf{1})]}{n^d}. \end{aligned}$$

## 6.2 Examples

### Example 1:

Here we consider a coisometric lifting for  $d = 1$ . Let

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix},$$

on  $\mathcal{H}_E = l^2(\mathbb{N}) \oplus l^2(\mathbb{Z})$ , be a lifting of  $C$  on  $\mathcal{H}_C = l^2(\mathbb{N})$ . We denote the standard basis for  $l^2(\mathbb{N})$  and  $l^2(\mathbb{Z})$  by  $\{e_i\}_{i=1}^\infty$  and  $\{g_i\}_{i=-\infty}^\infty$  respectively. Denote the shift operator on  $l^2(\mathbb{N})$  by  $S$ . Take operator  $C = S^*$ . Further, fix  $0 \leq \lambda \leq 1$  and define  $B$  and  $A$  as follows:

$$B\left(\sum_{i=1}^{\infty} a_i e_i\right) := \sqrt{1-\lambda^2} \ a_1 g_1,$$

$$A\left(\sum_{i=-\infty}^{\infty} a_i g_i\right) = \sum_{i=-\infty}^{\infty} c_i g_i,$$

where  $c_1 = \lambda a_0$  and  $c_{i+1} = a_i$  for  $i \neq 0$ .

It is easy to verify that  $CC^* = \mathbf{1}$ ,  $BC^* = 0$  and

$$BB^* + AA^* = \mathbf{1}.$$

This implies that  $E$  is coisometric. Here  $\mathbf{1} - C^*C$  is a projection onto  $\mathbb{C}e_1$ . In this case  $\mathcal{D}_{*,A} = \mathbb{C}g_1$  and  $\mathcal{D}_C = \mathbb{C}e_1$ . The isometry  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  is given by

$$\gamma(a_1g_1) = a_1e_1.$$

The characteristic function  $M_{C,E}$  of the lifting in this case maps  $l^2(\mathbb{N}) \otimes \mathcal{D}_E$  to  $l^2(\mathbb{N}) \otimes \mathcal{D}_C$  with symbol  $\theta_{C,E}$ . Let  $\omega$  denote the unit vector  $(1, 0, 0, \dots)$  of  $l^2(\mathbb{N})$  in  $l^2(\mathbb{N}) \otimes \mathcal{D}_C$ . Thus for  $h = \sum_{i=1}^{\infty} a_i e_i \in \mathcal{H}_C = l^2(\mathbb{N})$

$$\theta_{C,E}d_h = (\lambda^2 a_1 e_1)\omega,$$

and for  $h = \sum_{-\infty}^{\infty} a_i g_j \in \mathcal{H}_A = l^2(\mathbb{Z})$

$$\theta_{C,E}d_h = -(\lambda\sqrt{1-\lambda^2} a_0 e_1)\omega.$$

Note that here  $M_{C,E}$  and  $M_{J^s,C,E}$  will be the same.

We make use of the formula from Theorems C and D of [Ar03] to calculate

$$\begin{aligned} \text{curv}_s A &= \lim_{n \rightarrow \infty} \frac{\text{trace}(\mathbf{1} - A^{n+1}(A^*)^{n+1})}{n} = \lim_{n \rightarrow \infty} \frac{(n+1)(1-\lambda^2)}{n} = 1 - \lambda^2 \\ \chi_s(A) &= \lim_{n \rightarrow \infty} \frac{\text{rank}(\mathbf{1} - A^{n+1}(A^*)^{n+1})}{n} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1. \end{aligned}$$

**Example 2:**

Assume  $\underline{E} = (E_1, E_2)$  on  $\mathcal{H}_E = l^2(\mathbb{N}) \oplus \mathbb{C}f$  for a unit vector  $f$ , given by

$$E_i = \begin{pmatrix} C_i & 0 \\ B_i & A_i \end{pmatrix}$$

for  $i = 1, 2$ . The subspaces  $\mathcal{H}_C = l^2(\mathbb{N})$  and  $\mathcal{H}_A = \mathbb{C}f$  give the decomposition of  $\mathcal{H}_E$  relative to the above block matrix form. Denote the shift operator on  $l^2(\mathbb{N})$  by  $S$ . Take

$$C_1 = C_2 = \frac{1}{\sqrt{2}}S^*.$$

We fix a real number  $0 < t < 1$  and then take

$$B_1 = B_2 = \frac{1}{\sqrt{2}}(\sqrt{1-t^2}, 0, 0, \dots),$$

$$A_1 = A_2 = \frac{t}{\sqrt{2}}.$$

Clearly  $\underline{E}$  is a coisometric lifting of  $\underline{A}$ . Because  $\underline{A}$  is  $*$ -stable, this lifting is also subisometric. The defect space  $\mathcal{D}_{*,A} = \mathbb{C}f$ . The isometry  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  turns out to be

$$\gamma(kf) = \frac{k}{\sqrt{2}} \begin{pmatrix} (1, 0, 0, \dots)^T \\ (1, 0, 0, \dots)^T \end{pmatrix}$$

for  $k \in \mathbb{C}$ . Superscript  $T$  denotes transpose. Finally the constrained characteristic function is given for  $h = (h_1, h_2, \dots) \in \mathcal{H}_C$  by

$$\begin{aligned} \theta_{J^s, C, E} d_h^1 &= e_0 \otimes \begin{pmatrix} ((\frac{t^2+1}{2})h_1, -\frac{h_2}{2}, -\frac{h_3}{2}, \dots)^T \\ ((\frac{t^2-1}{2})h_1, -\frac{h_2}{2}, -\frac{h_3}{2}, \dots)^T \end{pmatrix} \\ &\quad - \sum_{|\alpha| \geq 1} L_\alpha^{J^s} e_0 \otimes \frac{(1-t^2)t^{|\alpha|}h_1}{2} \begin{pmatrix} (1, 0, 0, \dots)^T \\ (1, 0, 0, \dots)^T \end{pmatrix} \\ \theta_{J^s, C, E} d_h^2 &= e_0 \otimes \begin{pmatrix} ((\frac{t^2-1}{2})h_1, -\frac{h_2}{2}, -\frac{h_3}{2}, \dots)^T \\ ((\frac{t^2+1}{2})h_1, -\frac{h_2}{2}, -\frac{h_3}{2}, \dots)^T \end{pmatrix} \\ &\quad - \sum_{|\alpha| \geq 1} L_\alpha^{J^s} e_0 \otimes \frac{(1-t^2)t^{|\alpha|}h_1}{2} \begin{pmatrix} (1, 0, 0, \dots)^T \\ (1, 0, 0, \dots)^T \end{pmatrix} \end{aligned}$$

and for  $k \in \mathcal{H}_A$ ,  $i = 1, 2$  by

$$\begin{aligned} \theta_{J^s, C, E} d_k^i &= -e_0 \otimes \frac{t(\sqrt{1-t^2})k}{2} \begin{pmatrix} (1, 0, 0, \dots)^T \\ (1, 0, 0, \dots)^T \end{pmatrix} \\ &\quad + \sum_{\alpha} L_i^{J^s} L_\alpha^{J^s} e_0 \otimes \frac{t^{|\alpha|}(\sqrt{1-t^2})(1-\frac{t^2}{2})k}{\sqrt{2}} \begin{pmatrix} (1, 0, 0, \dots)^T \\ (1, 0, 0, \dots)^T \end{pmatrix} \\ &\quad + \sum_{j \neq i} \sum_{\alpha} L_j^{J^s} L_\alpha^{J^s} e_0 \otimes \frac{t^{|\alpha|+2}(\sqrt{1-t^2})k}{2\sqrt{2}} \begin{pmatrix} (1, 0, 0, \dots)^T \\ (1, 0, 0, \dots)^T \end{pmatrix} \end{aligned}$$



We calculate the invariants of the corresponding Hilbert module like in the previous example and as expected they are zero.

$$\begin{aligned} \text{curv}_s A &= 2! \lim_{n \rightarrow \infty} \frac{\text{trace}(\mathbf{1} - \sum_{|\alpha|=n+1} A_\alpha A_\alpha^*)}{n^2} = 2! \lim_{n \rightarrow \infty} \frac{1 - t^{2(n+1)}}{n^2} = 0 \\ \chi_s(A) &= 2! \lim_{n \rightarrow \infty} \frac{\text{rank}(\mathbf{1} - \sum_{|\alpha|=n+1} A_\alpha A_\alpha^*)}{n^2} = 2! \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0. \end{aligned}$$

**Example 3:**

The following example is for the noncommutative case. Let  $\underline{E}$  on  $\mathcal{H}_E = \mathbb{C} \oplus \Gamma(\mathbb{C}^2)$  be a coisometric lifting of  $\underline{C} = (1, 0)$  on  $\mathcal{H}_C = \mathbb{C}$ . We take  $B_1 = 0$  and  $B_2(k) = k e_0$ , and  $\underline{A} = (L_1, L_2)$ , i.e., the tuple of creation operators. Thus  $D_{*,A}$  is the projection onto the vacuum  $\mathbb{C}e_0$ . Clearly for  $h_1, h_2 \in \mathcal{H}_C$  we have  $D_C(h_1, h_2) = h_2$ . It follows that  $\gamma : \Gamma(\mathbb{C}^2) \rightarrow \mathbb{C}$  takes  $e_0$  to 1 and maps  $\Gamma(\mathbb{C}^2) \ominus \mathbb{C}e_0$  to 0. The characteristic function is zero. Finally using Equations (2.16) and (4.1) of [Po02] we get

$$\begin{aligned} \text{curv} A &= \lim_{m \rightarrow \infty} \frac{\text{trace}(\mathbf{1} - \sum_{|\alpha|=m} L_\alpha L_\alpha^*)}{2^m} = \lim_{m \rightarrow \infty} \frac{1 + 2 + 2^2 + \dots + 2^{m-1}}{2^m} = 1 \\ \chi(A) &= \lim_{m \rightarrow \infty} \frac{\text{rank}(\mathbf{1} - \sum_{|\alpha|=m} L_\alpha L_\alpha^*)}{2^m} = \lim_{m \rightarrow \infty} \frac{1 + 2 + 2^2 + \dots + 2^{m-1}}{2^m} = 1. \end{aligned}$$

Instead if one chooses  $\underline{A}$  to be  $S_\perp$  constructed in Theorem 3.5, 3.8 and 4.9 of [Po02], then one can realise any  $t$  in  $[0, 1]$  as curvature invariant and Euler characteristic.

## 6.3 Liftings of covariant representations of $W^*$ -correspondences and Hardy algebras

First we prefer to remark that any row contraction  $\underline{T} = (T_1, \dots, T_d)$  on a Hilbert space  $\mathcal{H}$  can be encoded as the covariant representation  $(T, \sigma)$  of

the  $W^*$ -correspondence  $\mathbb{C}^d$  over the von Neumann algebra  $\mathbb{C}$  on  $\mathcal{H}$ . In this picture if  $\{e_1, \dots, e_d\}$  denote the standard basis of  $\mathbb{C}^d$ , then  $T_i = T(e_i)$  for all  $i$ . In the current section we will generalize the theory of chapter 4 for covariant representations of  $W^*$ -correspondence. The constrained dilation theory corresponding to Cuntz-Kreiger constraints of the last chapter fits into this scheme but the  $q$ -commuting dilation theory does not. The reader may need to refer back to some notions defined in Section 2.4.

Let  $(C, \sigma_C)$  be a contractive covariant representation of  $\mathcal{E}$  on  $\mathcal{H}_C$ . Then a contractive covariant representation  $(E, \sigma_E)$  of  $\mathcal{E}$  on a Hilbert space  $\mathcal{H}_E \supset \mathcal{H}_C$  is called a *contractive lifting* of  $(C, \sigma_C)$  if

- (a)  $\mathcal{H}_C$  reduces  $\sigma_E$  and for all  $a \in \mathcal{M}$

$$\sigma_E(a)|_{\mathcal{H}_C} = P_{\mathcal{H}_C} \sigma_E(a)|_{\mathcal{H}_C} = \sigma_C(a).$$

- (b)  $\mathcal{H}_C^\perp$  is invariant w.r.t.  $E(\xi)$  for all  $\xi \in \mathcal{E}$ .

- (c)  $P_{\mathcal{H}_C} E(\xi)|_{\mathcal{H}_C} = C(\xi)$  for all  $\xi \in \mathcal{E}$ .

Set  $A(\xi) = P_{\mathcal{H}_C^\perp} E(\xi)|_{\mathcal{H}_C^\perp}$  and  $\sigma_A(a) := \sigma_E(a)|_{\mathcal{H}_C^\perp}$  for all  $\xi \in \mathcal{E}, a \in \mathcal{M}$ . Observe that  $(A, \sigma_A)$  is also a covariant representation of  $\mathcal{E}$ . This definition of contractive lifting is equivalent to assuming that  $\tilde{E}$  is contractive and has the form

$$\tilde{E} = \begin{pmatrix} \tilde{C} & 0 \\ \tilde{B} & \tilde{A} \end{pmatrix}.$$

Note that if  $(E, \sigma_E)$  is completely contractive then  $(C, \sigma_C)$  and  $(A, \sigma_A)$  are also completely contractive. This follows easily by passing to  $\tilde{E}$  and using Lemma 2.4.3.

**Definition 6.3.1.** Let  $(T, \sigma)$  be a completely contractive covariant (c.c.c. for short) representation of  $\mathcal{E}$  on  $\mathcal{H}$ . An isometric dilation  $(V, \pi)$  of  $(T, \sigma)$  is an isometric covariant representation of  $\mathcal{E}$  on  $\tilde{\mathcal{H}} \supset \mathcal{H}$  such that  $(V, \pi)$

is a lifting of  $(T, \sigma)$ . A minimal isometric dilation (*mid*) of  $(T, \sigma)$  is an isometric dilation  $(V, \pi)$  on  $\hat{\mathcal{H}}$  for which (as before)

$$\hat{\mathcal{H}} = \overline{\text{span}}\{V(\xi_1) \dots V(\xi_n)h : h \in \mathcal{H}, \xi_i \in \mathcal{E}\}.$$

Further one defines the *full Fock module* over  $\mathcal{M}$  as

$$\mathcal{F}(\mathcal{E}) = \oplus_{i=0}^{\infty} \mathcal{E}^{\otimes i}$$

where  $\mathcal{E}^{\otimes 0} = \mathcal{M}$ . We will write  $\mathcal{F}$  for  $\mathcal{F}(\mathcal{E})$  in short. For a  $\xi \in \mathcal{E}$  the creation operator  $L_\xi$  on  $\mathcal{F}(\mathcal{E})$  is given by  $L_\xi \eta = \xi \otimes \eta$ . We have an induced homomorphism  $\varphi^n$  from  $\mathcal{M}$  to  $\mathcal{L}(\mathcal{E}^{\otimes n})$  which for each  $a \in \mathcal{M}$  is given by

$$\varphi^n(a)(\xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \dots \otimes \xi_n.$$

Take the operator

$$\varphi_\infty(a) = \text{diag}(a, \varphi(a), \varphi^2(a), \dots)$$

on  $\mathcal{F}$ .

A mid of a c.c.c. representation always exist and is unique up to unitary equivalence (cf. [MS05]). We give a brief sketch of the proof: Given a c.c.c. representation  $(T, \sigma)$  of a  $W^*$ -correspondence  $\mathcal{E}$  on a Hilbert space  $\mathcal{H}$ , we consider the associated  $\tilde{T} : \mathcal{E} \otimes \mathcal{H} \rightarrow \mathcal{H}$ . We set  $D_{*,T} := (\mathbf{1} - \tilde{T}\tilde{T}^*)^{\frac{1}{2}}$  (in  $B(\mathcal{H})$ ) and  $D_T := (\mathbf{1} - \tilde{T}^*\tilde{T})^{\frac{1}{2}}$  (in  $B(\mathcal{E} \otimes_\sigma \mathcal{H})$ ). Let  $\mathcal{D}_{*,T} := \overline{\text{range } D_{*,T}}$  and  $\mathcal{D}_T = \overline{\text{range } D_T}$ . The space of mid  $(V, \pi)$  is

$$\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{F} \otimes_{\sigma_1} \mathcal{D}_T$$

where  $\sigma_1$  is defined to be the restriction to  $\mathcal{D}_T$  of  $\varphi(\cdot) \otimes \mathbf{1}$ . Finally a representation  $\pi$  of  $\mathcal{E}$  on  $\hat{\mathcal{H}}$  given by

$$\pi = \sigma \oplus \sigma_1^{\mathcal{F}} \circ \varphi_\infty$$

with  $(\sigma_1^{\mathcal{F}} \circ \varphi_\infty)(a) = \varphi_\infty(a) \otimes \mathbf{1}_{\mathcal{D}_T}$  for  $a \in \mathcal{M}$ , and a linear map  $V : \mathcal{E} \rightarrow B(\hat{\mathcal{H}})$  given in operator matrix form by:

$$V(\xi) = \begin{pmatrix} T(\xi) & 0 & 0 & \dots \\ D_T(\xi \otimes \cdot) & 0 & 0 & \dots \\ 0 & \mathbf{1} & 0 & \\ 0 & 0 & \mathbf{1} & \\ \vdots & & & \ddots \end{pmatrix}$$

together constitute a mid  $(V, \pi)$  of  $(T, \sigma)$ . □

Note that  $\tilde{V}$  will be the mid of  $\tilde{T}$ . Moreover if

$$\tilde{T}\tilde{T}^* = \mathbf{1},$$

then  $(T, \sigma)$  is said to be *coisometric*. It is also known that the mid  $(V, \pi)$  is coisometric if and only if  $(T, \sigma)$  is coisometric.

Let us define  $\tilde{T}^n = \tilde{T}(\mathbf{1} \otimes \tilde{T}^{n-1})$  for  $n \geq 1$ . We say  $(T, \sigma)$  is *\*-stable* if  $\lim_{n \rightarrow \infty} \tilde{T}^n(\tilde{T}^n)^* = 0$  in SOT.

The algebra defined below is intrinsically related to  $H^\infty$  which will become apparent when we go through the examples listed after it. Because the characteristic functions and Poisson kernel of Sz-Nagy and Foias are  $H^\infty$  functions, this new algebra will be crucial for extending our theory of chapter 4 for  $W^*$ -correspondences.

**Definition 6.3.2.** *The ultraweakly closed subalgebra of  $\mathcal{L}(\mathcal{F})$  generated by the  $L_\xi$ 's and  $\varphi_\infty(a)$ 's is called the Hardy algebra of  $\mathcal{E}$  and is denoted by  $H^\infty(\mathcal{E})$ .*

From works of Muhly and Solel ([MS05]) it is known that there are 1-1 correspondences:

- (a) between completely contractive covariant representations  $(T, \sigma)$  and contractive  $\tilde{T}$ 's.

- (b) between completely contractive  $(T, \sigma)$  and its integrated form  $\sigma \times T : H^\infty(\mathcal{E}) \rightarrow B(\mathcal{H})$ , if  $\|\tilde{T}\| < 1$ . Here  $\sigma \times T$  maps

$$\varphi_\infty(a) \mapsto \sigma(a), L_\xi \mapsto T(\xi).$$

**Lemma 6.3.3.**

$$\tilde{T}(\varphi(a) \otimes \mathbf{1}) = \sigma(a)\tilde{T} \quad \text{for } a \in \mathcal{M}.$$

Therefore  $\tilde{T}^*\tilde{T}$  commutes with  $(\varphi(\mathcal{M}) \otimes \mathbf{1})$  and  $\tilde{T}\tilde{T}^*$  commutes with  $\sigma(\mathcal{M})$ .

*Proof.* For  $\xi \otimes h \in \mathcal{H}^E$  and  $a \in \mathcal{M}$  we have

$$\begin{aligned} \tilde{T}(\varphi(a) \otimes \mathbf{1})(\xi \otimes h) &= \tilde{T}(\varphi(a)\xi \otimes h) \\ &= T(\varphi(a)\xi)h = \sigma(a)T(\xi)h \\ &= \sigma(a)\tilde{T}(\xi \otimes h). \end{aligned}$$

□

We remark that each element  $\mu \in \mathcal{E}^\sigma$  there is a representation  $(T, \sigma)$  of  $\mathcal{E}$  such that  $\tilde{T} = \mu^*$ . Using the above bijective relations we define for each  $G \in H^\infty(\mathcal{E})$  a function  $\hat{G}$  given by

$$\hat{G}(\mu^*) := (\sigma \times T)(G) \quad \forall \mu = \tilde{T}^* \in \mathbb{D}(\mathcal{E}^\sigma).$$

This  $\hat{G}$  is called the *Fourier transform* of  $G$ . Thus elements of the Hardy algebra  $H^\infty(\mathcal{E})$  can be realised as functions on unit ball  $\mathbb{D}(\mathcal{E}^\sigma)$  analogous to classical  $H^\infty$  functions.

Consider the following special case:

**Example 1:** When  $\mathcal{M} = \mathcal{E} = \mathbb{C}$  and  $\sigma$  is the identity representation of  $\mathcal{M}$  on  $\mathcal{H} = \mathbb{C}$ , then  $\mathbb{D}(\mathcal{E}^\sigma)$  is the open unit disc in the complex plane.

Any  $G \in \mathcal{H}^\infty(\mathcal{E})$  is basically an infinite, lower-triangular, Toeplitz matrix on  $l^2(\mathbb{N})$ :

$$G = \begin{pmatrix} a_0 & 0 & 0 & 0 & \dots \\ a_1 & a_0 & 0 & 0 & \dots \\ a_2 & a_1 & a_0 & 0 & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & & & \ddots \end{pmatrix}.$$

The Fourier transform  $\hat{G} : \mathbb{D}(\mathcal{E}^\sigma)^* \rightarrow B(\mathcal{H})$  is

$$\bar{\mu} \mapsto \sum_{i=0}^{\infty} a_i \bar{\mu}^i$$

with  $\sum_{i=0}^{\infty} a_i \bar{\mu}^i$  acting via multiplication on  $\mathcal{H} = \mathbb{C}$ .

**Example 2:** With the same  $\mathcal{M} = \mathcal{E} = \mathbb{C}$  as above but  $\mathcal{H}$  a Hilbert space instead, we take in this example the representation  $\sigma$  of  $\mathbb{C}$  on  $\mathcal{H}$  as multiplication, i.e.,

$$\sigma(c)h = ch, \quad c \in \mathbb{C}, h \in \mathcal{H}.$$

Then  $\mathcal{E}^\sigma \cong B(\mathcal{H})$  and  $\mathbb{D}(\mathcal{E}^\sigma)$  is the set of all strict contractions on  $\mathcal{H}$ .  $\mathcal{H}^\infty(\mathcal{E})$  is still the set of all lower-triangular Toeplitz matrices as before. If  $T \in \mathbb{D}(\mathcal{E}^\sigma)$ , then

$$\hat{G}(T^*) = \sum_{i=0}^{\infty} a_i (T^*)^i.$$

**Example 3:** Next, consider the case when  $\mathcal{M} = \mathbb{C}, \mathcal{E} = \mathbb{C}^n$  and take  $\sigma$  to be the representation of  $\mathcal{M}$  on  $\mathcal{H}$  same as in Example 2. The  $\mathbb{D}(\mathcal{E}^\sigma)$  is the set of all row contractions  $\underline{T}$  with norm  $\|\underline{T}\underline{T}^*\|$  less than 1.

We can also define the Poisson kernel in module context. For every  $\mu \in \mathbb{D}(\mathcal{E}^\sigma)$  we set an operator  $\mu^{(n)} : \mathcal{H} \rightarrow \mathcal{E}^{\otimes n} \otimes_\sigma \mathcal{H}$  by

$$\mu^{(n)} := (\mathbf{1}_{\mathcal{E}^{\otimes n-1}} \otimes \mu)(\mathbf{1}_{\mathcal{E}^{\otimes n-2}} \otimes \mu) \dots (\mathbf{1}_{\mathcal{E}} \otimes \mu)\mu$$

Now with it we associate the operator called *Poisson kernel*

$$K(\mu) := (\mathbf{1}_{\mathcal{F}} \otimes (\mathbf{1} - \mu^* \mu)^{\frac{1}{2}}) [\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \dots]^T.$$

which maps  $\mathcal{H}$  to  $\mathcal{F} \otimes_{\sigma} \mathcal{H}$ .

Characteristic functions of covariant representations of  $W^*$ -correspondences have been studied by Muhly and Solel (cf. [MS05]). Here we are interested in the corresponding theory for liftings of covariant representations. We consider two special cases of liftings as in the last chapter and then investigate the general case.

Let  $(V^E, \pi_E)$  and  $(V^C, \pi_C)$  denote the mids of  $(E, \sigma_E)$  and  $(C, \sigma_C)$  respectively. From the definition of lifting it is immediate that the space of the mid  $V^C$  can be embedded as a subspace reducing  $V^E$ .

**Definition 6.3.4.** *A lifting  $(E, \sigma_E)$  of a completely contractive covariant representation  $(C, \sigma_C)$  on  $\mathcal{H}_E \supset \mathcal{H}_C$  is called subisometric if the corresponding mids  $V^E$  and  $V^C$  are unitarily equivalent, i.e., there exists a unitary  $W : \hat{\mathcal{H}}_E \rightarrow \hat{\mathcal{H}}_C$  such that  $W|_{\mathcal{H}_C} = \mathbf{1}|_{\mathcal{H}_C}$ ,  $WV^E(\xi) = V^C(\xi)W$  for all  $\xi \in \mathcal{E}$  and  $W\pi_E(a) = \pi_C(a)W$  for all  $a \in \mathcal{M}$ .*

**Remark 6.3.5.** *Alternatively, a subisometric lifting means the existence of a unitary  $W$  (same as above) such that*

$$\tilde{V}^C(1 \otimes W) = W\tilde{V}^E.$$

**Proposition 6.3.6.** *Let  $(C, \sigma_C)$  be a completely contractive covariant (c.c.c. for short) representation of  $W^*$ -correspondence  $\mathcal{E}$  on  $\mathcal{H}_C$ . A completely contractive lifting  $(E, \sigma_E)$  on  $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$  of  $(C, \sigma_C)$  with*

$$E(\xi) = \begin{pmatrix} C(\xi) & 0 \\ B(\xi) & A(\xi) \end{pmatrix}, \quad \xi \in \mathcal{E},$$

*is subisometric if and only if  $(A, \sigma_A)$  is  $*$ -stable and  $\tilde{B} = D_{*,A}\gamma^*D_C$  with an isometry  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$ .*

The *characteristic function* of subisometric lifting as before is defined as

$$M_{C,E} := W|_{\mathcal{F} \otimes \mathcal{D}_E}.$$

**Theorem 6.3.7.** *Let  $(C, \sigma_C)$  be a c.c.c. representation of  $\mathcal{E}$  on a Hilbert space  $\mathcal{H}_C$ . Two subisometric liftings  $(E, \sigma_E)$  and  $(E', \sigma_{E'})$  of  $(C, \sigma_C)$  are unitarily equivalent if and only if the corresponding characteristic functions  $M_{C,E}$  and  $M_{C,E'}$  are unitarily equivalent.*

*Proof.* For the proof of the necessary part we assume that the liftings  $(E, \sigma_E)$  and  $(E', \sigma_{E'})$  are c.c.c. representations on  $\mathcal{H}_E$  and  $\mathcal{H}_{E'}$ , and  $U : \mathcal{H}_E \rightarrow \mathcal{H}_{E'}$  is a unitary such that  $U|_{\mathcal{H}_C} = \mathbf{1}_{\mathcal{H}_C}$  and

$$UE(\xi) = E'(\xi)U, \quad U\sigma_E(a) = \sigma_{E'}(a)U \quad \forall \xi \in \mathcal{E}, a \in \mathcal{M},$$

The mids of unitarily equivalent row contractions are unitarily equivalent. Hence we extend  $U$  canonically to a unitary  $\hat{U} : \hat{\mathcal{H}}_E \rightarrow \hat{\mathcal{H}}_{E'}$  defined between the spaces of mids  $(V^E, \pi_E)$  and  $(V^{E'}, \pi_{E'})$  with  $\hat{U}|_{\mathcal{H}_E} = U$ , and we get

$$\hat{U}V^E(\xi) = V^{E'}(\xi)\hat{U}, \quad \hat{U}\pi_E(a) = \pi_{E'}(a)\hat{U} \quad \forall \xi \in \mathcal{E}, a \in \mathcal{M}$$

Because  $(E, \sigma_E)$  and  $(E', \sigma_{E'})$  are subisometric we also have unitaries  $W : \hat{\mathcal{H}}_E \rightarrow \hat{\mathcal{H}}_C$  and  $W' : \hat{\mathcal{H}}_{E'} \rightarrow \hat{\mathcal{H}}_C$  respectively with:

$$\begin{aligned} V^C(\xi)W &= WV^E(\xi), & W|_{\mathcal{H}_C} &= \mathbf{1}_{\mathcal{H}_C} \\ V^C(\xi)W' &= W'V^{E'}(\xi), & W'|_{\mathcal{H}_C} &= \mathbf{1}_{\mathcal{H}_C} \end{aligned}$$

Let us take

$$U_C := W'\hat{U}W^* : \hat{\mathcal{H}}_C \rightarrow \hat{\mathcal{H}}_C.$$

Chasing a commuting diagram similar to diagram 4.24 and arguing on the lines of Theorem 4.1.6 we find that the characteristic functions are equivalent.



Conversely we show that if  $\Theta = \Theta'V$  with a unitary  $V : \mathcal{D}_E \rightarrow \mathcal{D}_{E'}$  then the two subisometric liftings  $(E, \sigma_E)$  and  $(E', \sigma_{E'})$  are unitarily equivalent. Let  $W$  and  $W'$  be the corresponding unitaries from the subisometric lifting property. Then

$$\begin{aligned} W'\mathcal{H}_{E'} &= \mathcal{H}_C \oplus (\mathcal{F} \otimes \mathcal{D}_C) \ominus W'(\mathcal{F} \otimes \mathcal{D}_{E'}) \\ &= \mathcal{H}_C \oplus (\mathcal{F} \otimes \mathcal{D}_C) \ominus M_{C,E'}(\mathcal{F} \otimes V\mathcal{D}_E) \\ &= \mathcal{H}_C \oplus (\mathcal{F} \otimes \mathcal{D}_C) \ominus M_{C,E}(\mathcal{F} \otimes \mathcal{D}_E) = W\mathcal{H}_E, \end{aligned}$$

and we can define

$$U := (W')^* W|_{\mathcal{H}_E} : \mathcal{H}_E \rightarrow \mathcal{H}_{E'}.$$

Because for  $h \in \mathcal{H}_C$ ,  $Wh = h = W'h$  we have  $Uh = h$ . In general for  $h \in \mathcal{H}_E$  and  $\xi \in \mathcal{E}$  (with  $p_E, p_{E'}$  orthogonal projections onto  $\mathcal{H}_E, \mathcal{H}_{E'}$ )

$$\begin{aligned} UE(\xi)h &= (W')^* W E(\xi)h = (W')^* W p_E V^E(\xi)h = p_{E'} (W')^* W V^E(\xi)h \\ &= p_{E'} (W')^* V^C(\xi)Wh = p_{E'} V^{E'}(\xi)(W')^* Wh = E'(\xi)Uh. \end{aligned}$$

Identical computation gives

$$U\sigma_E(a) = \sigma_{E'}(a)U \quad \text{for } a \in \mathcal{M}.$$

Hence  $\underline{E}$  and  $\underline{E}'$  are unitarily equivalent.  $\square$

Consider the coisometric liftings of  $(C, \sigma_C)$  by  $*$ -stable  $(A, \sigma_A)$ . Then the unitary equivalence classes of those which are also subisometric liftings of  $(C, \sigma_C)$  are parametrized by isometries  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$ .

Next we deal with the general case where  $(E, \sigma_E)$  is a contractive lifting of  $(C, \sigma_C)$ . Because of the structure of lifting it is immediate that the space of mid  $(V^C, \pi_C)$  is embedded in  $(V^E, \pi_E)$ . We introduce a c.c.c.

representation  $(Y, \pi_Y)$  on the orthogonal complement  $\mathcal{K}$  of the space of mid  $(V^C, \pi_C)$  to encode this. Hence we can get a unitary  $W$  such that

$$W : \mathcal{H}_E \oplus (\mathcal{F} \otimes \mathcal{D}_E) \rightarrow \mathcal{H}_C \oplus (\mathcal{F} \otimes \mathcal{D}_C) \oplus \mathcal{K},$$

$$\hat{V}^E(\xi)W = WV^E(\xi), \quad W|_{\mathcal{H}_C} = \mathbf{1}|_{\mathcal{H}_C} \quad \text{with} \quad \hat{V}^E(\xi) = V^C(\xi) \oplus Y(\xi).$$

Recall that  $\mathcal{F}$  denote the full Fock module on  $\mathcal{E}$ . We denote by the same  $W$  its restriction to the complement of  $\mathcal{H}_C$  too, i.e.,

$$W : \mathcal{H}_A \oplus (\mathcal{F} \otimes \mathcal{D}_E) \rightarrow (\mathcal{F} \otimes \mathcal{D}_C) \oplus \mathcal{K}$$

With this we get

$$\begin{aligned} B(\xi)^* &= p_{\mathcal{H}_C} V^E(\xi)^*|_{W\mathcal{H}_A} = p_{\mathcal{H}_C} [V^C(\xi)^* \oplus Y(\xi)^*]W|_{\mathcal{H}_A} \\ &= (D_C(\xi \otimes \cdot))^* p_{\mathcal{D}_C} W|_{\mathcal{H}_A}. \end{aligned}$$

Doing computations on the lines of those appearing in Section 4.3 we obtain

$$P_{\mathcal{D}_C} W|_{\mathcal{H}_A} = \gamma D_{*,A}.$$

We have shown that if  $(E, \sigma_E)$  is a c.c.c. lifting of  $(C, \sigma_C)$  by  $(A, \sigma_A)$  as above then

$$\tilde{B} = D_{*,A} \gamma^* D_C. \tag{6.3}$$

For the converse we start with two c.c.c. representations  $(C, \sigma_C)$  and  $(A, \sigma_A)$ , a contraction  $\gamma : \mathcal{D}_{*,A} \rightarrow \mathcal{D}_C$  and  $\tilde{B}$  as in equation (5.9). Then for  $x \in \mathcal{H}_C$ ,  $y \in \mathcal{H}_A$

$$|\langle x, \tilde{C} \tilde{B}^* y \rangle|^2 = |\langle x, \tilde{C} D_C^* \gamma D_{*,A} y \rangle|^2 = |\langle D_C \tilde{C}^* x, \gamma D_{*,A} y \rangle|^2$$

$$\leq \|D_C \tilde{C}^* x\|^2 \|\gamma D_{*,A} y\|^2 \leq \langle x, (\mathbf{1} - \tilde{C} \tilde{C}^*) x \rangle \langle y, (\mathbf{1} - \tilde{A} \tilde{A}^*) y \rangle.$$

As in Exercise 3.2 in [Pau03] it follows that

$$0 \leq \begin{pmatrix} \mathbf{1} - \tilde{C}\tilde{C}^* & -\tilde{C}\tilde{B}^* \\ -\tilde{B}\tilde{C}^* & \mathbf{1} - \tilde{A}\tilde{A}^* \end{pmatrix} = \mathbf{1} - \tilde{E}\tilde{E}^*.$$

Thus  $\tilde{E}$  is a contraction. We summarize this in the following lemma:

**Lemma 6.3.8.** *Let  $(E, \sigma_E)$  be a lifting of  $(C, \sigma_C)$  by  $(A, \sigma_A)$ . Then  $(E, \sigma_E)$  is completely contractive if and only if  $(C, \sigma_C)$  and  $(A, \sigma_A)$  are completely contractive and there exists a contraction  $\gamma$*

$$\tilde{B} = D_{*,A}\gamma^*D_C.$$

It turns out that

$$(\mathcal{F} \otimes \mathcal{D}_C) \vee W(\mathcal{F} \otimes \mathcal{D}_E) = (\mathcal{F} \otimes \mathcal{D}_C) \oplus \mathcal{K},$$

if

1.  $\gamma$  is *resolving*, i.e., for  $h \in \mathcal{H}_A$

$$\begin{aligned} &(\gamma D_{*,A}(A(\xi))^*h = 0 \text{ for all } \xi \in \mathcal{E}) \Rightarrow \\ &(D_{*,A}(A(\xi))^*h = 0 \text{ for all } \xi \in \mathcal{E}), \text{ and} \end{aligned}$$

2.  $(A, \sigma_A)$  is *c.n.c.*, i.e.,  $\mathcal{H}_A^1 := \{h \in \mathcal{H}_A : \|(\tilde{A}^n)^*h\| = \|h\| \text{ for all } n \in \mathbb{N}\} = \{0\}$ .

We call such liftings *reduced liftings*.

In this case the characteristic function of lifting is defined as

$$M_{C,E} = P_{\mathcal{F} \otimes \mathcal{D}_C} W|_{\mathcal{F} \otimes \mathcal{D}_E}.$$

It can be shown as in the Chapter 4 that for any completely contractive covariant representation  $(C, \sigma_C)$ , these characteristic functions are complete invariants for reduced liftings of  $(C, \sigma_C)$  up to unitary equivalence.

We set  $\mu = \tilde{A}^*$  and  $d_h^\xi := (V^E(\xi) - E(\xi))h = D_E(\xi \otimes h)$  for  $\xi \in \mathcal{E}$  and  $h \in \mathcal{H}_E$ . The expanded form of the characteristic function is the following:

**Case I:**  $h \in \mathcal{H}_C$ .

$$\Theta_{C,E}(d_h^\xi) = [D_C(\xi \otimes h) - \gamma D_{*,A}B(\xi)h] - \sum_{j=1}^{\infty} \gamma D_{*,A}(L_\xi^* \tilde{A}^*)^j B(\xi)h.$$

Alternatively

$$\Theta_{C,E}(d_h^\xi) = [D_C(\xi \otimes h) - \gamma D_{*,A}B(\xi)h] - \sum_{j=1}^{\infty} (\mathbf{1} \otimes \gamma D_{*,A})(\mathbf{1} \otimes \mu^j)(\mathbf{1} \otimes B(\xi)h).$$

**Case II:**  $h \in \mathcal{H}_A$ .

$$\theta_{C,E}(d_h^\xi) = -\gamma \tilde{A} d_h^\xi + \sum_{j=0}^{\infty} \gamma D_{*,A}(L_\xi^* \tilde{A}^*)^j L_\xi^* D_A d_h^\xi.$$

Alternatively

$$\theta_{C,E}(d_h^\xi) = -\gamma \mu^* d_h^\xi + \sum_{j=0}^{\infty} (\mathbf{1} \otimes \gamma D_{*,A})(\mathbf{1} \otimes \mu^j) D_A d_h^\xi.$$

Let us briefly mention one potential good application of this theory to analytic crossed products of the type  $\mathcal{M} \rtimes \mathbb{Z}_+$  (cf. Section 6 of [MS05]). Muhly and Solel showed in this last quoted work that one can associate a contraction  $t$  to any  $(\sigma$ -weakly continuous) representation of this crossed product. When  $t$  is c.n.c., its Sz. Nagy- Foias characteristic function is unitarily equivalent to the characteristic function of the covariant representation associated to this representation of  $\mathcal{M} \rtimes \mathbb{Z}_+$ . This theory needs to be explored for liftings of covariant representations.

## 6.4 Appendix

We list some facts about Hilbert  $C^*$ -modules and some associated important  $C^*$ -algebras. Given a Hilbert  $C^*$ -bimodule  $\mathcal{E}$  on a  $C^*$ -algebra  $\mathcal{A}$  with the associated left action of  $\mathcal{A}$  denoted by  $\varphi$ , the *Toeplitz  $C^*$ -algebra*  $\mathcal{T}(\mathcal{E})$  of  $\mathcal{E}$  is the  $C^*$ -subalgebra generated by  $\{L_\xi\}_{\xi \in \mathcal{E}}$  and  $\{\varphi_\infty(a)\}_{a \in \mathcal{A}}$  of  $\mathcal{L}(\mathcal{E})$ .

**Theorem 6.4.1.** (*Pimsner*) *If  $\mathcal{E}$  is a Hilbert  $C^*$ -bimodule on  $\mathcal{A}$ , then there is a bijective map from the set of all isometric covariant representations  $(V, \sigma)$  from  $\mathcal{E}$  on  $\mathcal{H}$  to the set of all  $C^*$ -representation from  $\mathcal{T}(\mathcal{E})$  on  $\mathcal{H}$ , determined by*

$$L_\xi \mapsto V(\xi), \quad \varphi_\infty(a) \mapsto \sigma(a).$$

An ideal  $\mathcal{I}$  in a  $C^*$ -algebra  $\mathcal{C}$  is called *essential* if there is no nonzero ideal of  $\mathcal{C}$  that has zero intersection with  $\mathcal{I}$ . For any algebra  $\mathcal{A}$  there always exists a unique (up to isomorphism) maximal  $C^*$ -algebra that contains  $\mathcal{A}$  as an essential ideal. This maximal ideal is called the *multiplier algebra* of  $\mathcal{A}$ , denoted by  $M(\mathcal{A})$ . Set  $\mathcal{B}$  as the  $C^*$ -subalgebra of  $\mathcal{L}(\mathcal{F}(\mathcal{E}))$  generated by  $\mathcal{L}(\sum_{n=0}^N \mathcal{E}^{\otimes n})$  for all  $N \in \mathbb{N}$ .

**Definition 6.4.2.** *A Cuntz-Pimsner algebra  $\mathcal{O}(\mathcal{E})$  is the image of  $\mathcal{T}(\mathcal{E})$  (under canonical embedding) in  $M(\mathcal{B})/\mathcal{B}$ .*

The Cuntz-Pimsner algebra generalizes Cuntz-Krieger algebra and cross-product of any  $C^*$ -algebra by  $\mathbb{Z}$ . The following are some examples of Cuntz-Pimsner algebras:

- (a) For  $\mathcal{A} = \mathcal{E} = \mathbb{C}$  we can identify the Fock module  $\mathcal{F}(\mathcal{E})$  with  $l^2(\mathbb{N})$ . Hence  $\mathcal{T}(\mathcal{E}) = C^*(L)$  where  $L$  is the unilateral shift and

$$\mathcal{O}(\mathcal{E}) \cong C(\mathbb{T})$$

where  $\mathbb{T}$  denotes the unit circle in the complex plane.

- (b) Now take  $\mathcal{A}$  to be any unital  $C^*$ -algebra with an automorphism  $\varphi$ . Then  $\mathcal{A}$  has a Hilbert module structure  $\mathcal{E}$  with the left action  $\tilde{\varphi} : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{E})$  is

$$a \mapsto \varphi(a)$$

Here  $\varphi(a)$ s are realised as elements of  $\mathcal{L}(\mathcal{E})$  acting as multiplication operator. We get  $\mathcal{O}(\mathcal{E}) \cong \mathcal{A} \times_{\varphi} \mathbb{Z}$ .

- (c) The Cuntz algebra  $\mathcal{O}_d$  can be realised as  $\mathcal{O}(\mathcal{E})$  for  $\mathcal{A} = \mathbb{C}$ ,  $\mathcal{E} = \mathbb{C}^d$  and

$$\varphi(k)\xi = (k.1)\xi \quad \text{for } k \in \mathcal{A}, \xi \in \mathcal{E}.$$

From a separable infinite dimensional Hilbert space  $\mathcal{H}$  and a  $C^*$ -algebra  $\mathcal{A}$  we can construct an important example of Hilbert module  $\mathcal{H}_{\mathcal{A}} := \mathcal{H} \otimes \mathcal{A}$  where the tensor product is given by

$$\langle h_1 \otimes a_1, h_2 \otimes a_2 \rangle = \langle h_1, h_2 \rangle a_1^* a_2 \quad \text{for } h_1, h_2 \in \mathcal{H}; a_1, a_2 \in \mathcal{A}.$$

Bellow some results are quoted which we need for Theorem 2.4.3. They show the usefulness of  $\mathcal{H}_{\mathcal{A}}$ .

**Theorem 6.4.3.** (*Kasparov's stabilisation theorem*) *If  $\mathcal{E}$  is a countably generated Hilbert  $\mathcal{A}$ -module, then  $\mathcal{E} \oplus \mathcal{H}_{\mathcal{A}} \cong \mathcal{H}_{\mathcal{A}}$ .*

**Corollary 6.4.4.** *If  $\mathcal{E}$  is a countably generated Hilbert  $\mathcal{A}$ -module, then the  $C^*$ -algebra of compacts  $K(\mathcal{E})$  is  $\sigma$ -unital, i.e.,  $K(\mathcal{E})$  has a countable approximate unit.*

# Bibliography

- [Ar69] Arveson, W.: *Subalgebras of  $C^*$ -algebras*, Acta Math., **123** (1969), 141-224.
- [Ar72] Arveson, W.: *Subalgebras of  $C^*$ -algebras II*, Acta Math., **128** (1972), 271-308.
- [Ar98] Arveson, W.: *Subalgebras of  $C^*$ -algebras III, Multivariable operator theory*, Acta Math., **181** (1998), no. 2, 159-228.
- [Ar00] Arveson, W.: *The curvature invariant of a Hilbert module over  $C[z_1, \dots, z_d]$* , J. Reine Angew. Math., **522** (2000), 173-236.
- [Ar02] Arveson, W.: *The Dirac operator of a commuting  $d$ -tuple*, J. Funct. Anal., **189** (2002), no. 1, 53-79.
- [Ar03] Arveson, W.: *Noncommutative Dynamics and  $E$ -Semigroups*, Springer Monographs in Mathematics (2003).
- [Bh96] Bhat, B.V.R.: *An index theory for quantum dynamical semigroups*, Trans. Amer. Math. Soc., **348** (1996), no. 2, 561-583.
- [BB02] Bhat, B.V.R.; Bhattacharyya, T.: *A model theory for  $q$ -commuting contractive tuples*, J. Operator Theory, **470** (2002), no. 1, 97-116.

- [BBD04] Bhat, B.V.R.; Bhattacharyya, T.; Dey, S.: *Standard noncommuting and commuting dilations of commuting tuples*, Trans. Amer. Math. Soc., **356** (2004), 1551-1568.
- [BDH88] Baillet, M.; Denizeau, Y.; Havet, J-F.: *Indice d'une esperance conditionnelle*, Compositio Math. **66** (1988).
- [BDZ06] Bhat, B.V.R.; Dey, S.; Zacharias, J.: *Minimal Cuntz-Krieger dilations and representations of Cuntz-Krieger algebras*, Proc. Indian Acad. Sci. Math. Sci., **116** (2006), no. 2, 193-220.
- [BES05] Bhattacharyya, T.; Eschmeier, J.; Sarkar, J.: *Characteristic function of a pure commuting contractive tuple*, Integral Equations Operator Theory, **53** (2005), no. 1, 23-32.
- [BP94] Bhat, B.V.R.; Parthasarathy, K.R.: *Kolmogorov's existence theorem for Markov processes in  $C^*$  algebras*, Proc. Indian Acad. Sci., Math. Sci., **104** (1994), 253-262.
- [Ber88] Bercovici, H.: *Operator Theory and Arithmetic in  $H^\infty$* , AMS, Math. Surveys and Monographs **26** (1988).
- [BBL04] Barreto, S.D.; Bhat, B.V.R.; Liebscher, V.; Skeide, M.: *Type I product systems of Hilbert modules*, J. Funct. Anal. **212** (2004), no. 1, 121-181.
- [BJKW00] Bratteli, O.; Jorgensen, P.; Kishimoto, A.; Werner, R.F.: *Pure states on  $\mathcal{O}_d$* , J. Operator Theory, **43** (2000), 97-143.
- [BJP96] Bratteli, O.; Jorgensen, P.E.T.; Price, G.L.: *Endomorphisms of  $B(\mathcal{H})$* , Quantization, nonlinear partial differential equations, and operator algebra (Cambridge, MA, 1994), 93-138, Proc. Sympos. Pure Math., **59**, Amer. Math. Soc., Providence, RI (1996).



- [Bu84] Bunce, J.W.: *Models for  $n$ -tuples of noncommuting operators*, J. Funct. Anal., **57** (1984), 21-30.
- [BT07] Benhida, C.; Timotin, D.: *Characteristic functions for multicontractions and automorphisms of the unit ball*, Integral Equations and Operator Theory, **57** (2007), no. 2, 153-166.
- [Ch74] Choi, M.D.: *A Schwarz inequality for positive linear maps on  $C^*$ -algebras*, Illinois J.Math., **18** (1974), 565-574.
- [Cu77] Cuntz, J.: *Simple  $C^*$ -algebras generated by isometries*, Commun. Math. Phys., **57** (1977), 173-185.
- [CK80] Cuntz, J.; Krieger, W.: *A class of  $C^*$ -algebras and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 251-268.
- [Da96] Davidson, K.R.:  *$C^*$ -algebras by example*, Fields Institute Monographs, 6, American Mathematical Society, Providence, RI, (1996).
- [De07a] Dey, S.: *Standard dilations of  $q$ -commuting tuples*, Colloq. Math., **107** (2007), no. 1, 141-165.
- [De07b] Dey, S.: *Constrained liftings and Hilbert modules* preprint (2007).
- [De08] Dey, S.: *Hardy algebras and liftings of covariant representations* preprint (2008).
- [Dr78] Drury, S.W.: *A generalization of von Neumann's inequality to the complex ball*, Proc. Amer. Math. Soc., **68** (1978), no. 3, 300-304.
- [DF84] Douglas, R.G.; Foias, C.: *Subisometric dilations and the commutant lifting theorem*, Topics in operator theory systems and networks, Workshop Rehovot/Isr. 1983, Oper. Theory, Adv. Appl., **12** (1984), 129-139.

- [DG07a] Dey, S.; Gohm, R.: *Characteristic Functions for Ergodic Tuples*, Integral Equations and Operator Theory, **58** (2007), 43-63
- [DG07b] Dey, S.; Gohm, R.: *Characteristic functions of liftings and applications for completely positive maps*, to appear in Journal of Operator Theory.
- [DKS01] Davidson, K.R.; Kribs, D.W.; Shpigel, M.E.: *Isometric dilations of non-commuting finite rank  $n$ -tuples*, Canad. J. Math. **53** (2001), 506-545.
- [ER00] Effros, E.G.; Ruan, Z.-J.: *Operator spaces*, London Mathematical Society Monographs. New Series, 23, The Clarendon Press, Oxford University Press, New York, (2000).
- [Fr82] Frazho, A.E.: *Models for noncommuting operators*, J. Funct. Anal., **48** (1982), 1-11.
- [FF90] Foias, C.; Frazho, A.E.: *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, 44 Birkhäuser Verlag, Basel (1990).
- [FNW94] Fannes, M.; Nachtergaele, B.; Werner, R.F.: *Finitely correlated states*, J.Funct.Anal., **120** (1994), 511-534.
- [GRS02] Greene, D.C.V.; Richter, S.; Sundberg, C.: *The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels*, J. Funct. Anal., **194** (2002), no. 2, 311–331.
- [Go04] Gohm, R.: *Noncommutative stationary processes*, Lecture Notes in Mathematics, 1839, Springer-Verlag, Berlin (2004).

- [Go06] Gohm, R.: *Decomposition of Beurling type for  $E_0$ -semigroups*, Quantum probability, Banach Center Publ., **73**, Polish Acad. Sci., Warsaw, (2006), 167-176.
- [GKL06] Gohm, R.; Kümmerer, B.; Lang, T.: *Non-commutative symbolic coding*, Ergod. Th. Dynam. Sys. **26**, (2006), 1-28.
- [Ka70] Kaplansky, I.: *Commutative rings*, Allyn and Bacon, Inc., Boston, Mass. (1970).
- [Kü85] Kümmerer, B.: *Markov dilations on  $W^*$ -algebras*, J. Funct. Anal., **63** (1985), 139-177.
- [Kü03] Kümmerer, B.: *Stationary Processes in Quantum Probability*, in S. Attal, M. Lindsay (Eds.), Quantum Probability Communications XI, World Scientific (2003), 273 - 304.
- [Kr01] Kribs, D.W.: *The curvature invariant of a non-commuting  $n$ -tuple*, Integral Equations Operator Theory, **41** (2001), no. 4, 426-454.
- [KM00] Kümmerer, B.; Maassen, H.: *A scattering theory for Markov chains*, Inf. Dim. Analysis, Quantum Prob. and Related Topics, vol. **3** (2000), 161-176.
- [La95] Lance, E. C.: *Hilbert  $C^*$ -modules. A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series, 210, Cambridge University Press, Cambridge, 1995.
- [MS98] Muhly, P.; Solel, B.: *Tensor Algebras over  $C^*$ -Correspondences: Representations, Dilations, and  $C^*$ -Envelopes*, J. Funct. Anal., **158** 389-457 (1998).
- [MS05] Muhly, P.; Solel, B.: *Canonical Models for Representations of Hardy Algebras*, Int. Eq. Oper. Theory, **53** (2005).

- [MS07] Muhly, P.; Solel, B.: The Poisson Kernel for Hardy Algebras, preprint (2007).
- [NF70] Sz.-Nagy, B.; Foias, C.: *Harmonic analysis of operators on Hilbert space*, North Holland Publ., Amsterdam-Budapest (1970).
- [Pa92] Parthasarathy, K.R.: *An introduction to quantum stochastic calculus*, Monographs in Mathematics, 85, Birkhäuser Verlag, Basel, 1992.
- [Pau03] Paulsen, V.I.: *Completely bounded maps and operator algebras*, Cambridge University Press (2003).
- [Pis01] Pisier, G.: *Similarity problems and completely bounded maps*, Second edition, Lecture Notes in Mathematics, 1618, Springer-Verlag, Berlin, (2001).
- [Pim97] Pimsner, M.V.: *A class of  $C^*$ -algebras generalizing both Cuntz-Krieger algebras and crossed products by  $\mathbb{Z}$* , Free probability theory (Waterloo, ON, 1995), 189-212, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, (1997).
- [Po89a] Popescu, G.: *Isometric dilations for infinite sequences of non-commuting operators*, Trans. Amer. Math. Soc., **316** (1989), 523-536.
- [Po89b] Popescu, G.: *Characteristic functions for infinite sequences of noncommuting operators*, J. Operator Theory, **22** (1989), 51-71.
- [Po89c] Popescu, G.: *Multi-analytic operators and some factorization theorems*, Indiana Univ.Math.J., **36** (1989), 693-710.
- [Po91] Popescu, G.: *von Neumann inequality for  $(B(\mathcal{H})^n)_1$* , Math. Scand. **68** (1991), no. 2, 292-304.

- [Po95] Popescu, G.: *Multi-analytic operators on Fock spaces*, J. Funct. Anal., **161** (1999), 27-61.
- [Po99] Popescu, G. : *Poisson transforms on some  $C^*$ -algebras generated by isometries*, J. Funct. Anal., **161** (1999), no. 1, 27-61.
- [Po02] Popescu, G.: *Curvature invariant for Hilbert modules over free semigroup algebras*, Adv. Math. **158**(2001), no. 2, 264-309.
- [Po03] Popescu, G.: *Similarity and ergodic theory of positive linear maps*, J. Reine Angew. Math., **561** (2003), 87-129.
- [Po05] Popescu, G.: *Operator theory on noncommutative varieties*, Indiana Univ. Math. J., **55** (2006), no. 2, 389-442.
- [Po06] Popescu, G.: *Free holomorphic functions on the unit ball of  $B(\mathcal{H})^n$* , J. Funct. Anal., **241** (2006), no. 1, 268-333.
- [Po07] Popescu, G.: *Operator Theory on noncommutative varieties-II*, Proc. Amer Math. Soc., **135** (2007), no. 7, 2151-2164.
- [Pow88] Powers, R.T.: *An index theory for semigroups of  $*$ -endomorphisms of  $B(\mathcal{H})$  and type  $II_1$  factors*, Canad. J. Math., **40** (1988), no. 1, 86-114.
- [Sk00] Skeide, M.: *Quantum stochastic calculus on full Fock modules*. J. Funct. Anal., **173** (2000), no. 2, 401-452.
- [Sk05a] Skeide, M.: *von Neumann modules, intertwiners and self-duality*, J. Operator Theory **54** (2005), no. 1, 119-124.
- [Sk05b] Skeide, M.: *Three ways to representations of  $B^A(E)$*  Quantum probability and infinite dimensional analysis, 504-517, QP-PQ: Quantum Probab. White Noise Anal., **18** World Sci. Publ., Hackensack, NJ, 2005.

- [Sk08] Skeide, M.: *Isometric dilations of representations of product systems via commutants*, Internat. J. Math. **19** (2008), no. 5, 521-539.
- [Ta01] Takesaki, M.: *Theory of operator algebras 1*, Springer (2001).